Output feedback distributed model predictive control with inherent robustness properties

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Abstract— **We consider robust output feedback distributed model predictive control (DMPC). The proposed controller is based on the results in [8] in which nominal stability and feasibility was proven for a DMPC-formulation without terminal constraint set or terminal cost in the optimization. We extend these results to show robust stability under state feedback as well as output feedback when dynamics and measurements are affected by bounded noise. The provided numerical example suggests that the region of attraction without terminal constraint set may be significantly larger than if a terminal constraint set is used.**

I. Introduction

In the model predictive control (MPC) literature nominal stability of the closed loop system is a well studied subject and is usually proven using a terminal constraint set, a terminal cost and a terminal controller, see [13] for a survey of such methods. Also robustness properties in MPC has received increased attention. In [13] different approaches from the literature to achieve robustness are presented. The survey shows three main approaches to guarantee robustness in MPC: to exploit the inherent robustness in nominal MPC, to design the MPC-controller to deal with any possible realization of the disturbance, or to introduce feedback in the design that compensates for the disturbances. Within the first category, it was shown in [9] that linear systems with convex constraints are inherently robust to small disturbances. This is due to the fact that the value function of the optimization problem is continuous [2], [9]. To address both robust feasibility and robust stability, a tube-based model predictive controller for linear systems was presented in [14]. This was extended to tube-based output feedback model predictive control in [12]. These tube-based MPC controllers also rely on a terminal cost and terminal constraints to show stability.

It was pointed out in [8] that terminal costs, terminal constraint sets, and terminal controllers usually involve all decision variables and are therefore not directly applicable for distributed model predictive control formulations where a centralized optimization problem is solved in distributed fashion. This is circumvented in [3] where stability is proven by setting a terminal point constraint in the origin, which is not desirable for performance and region of attraction reasons. In [8] a DMPC controller based on an optimization problem without terminal constraint set or terminal cost is proposed. Nominal stability for this is shown based on a controllability assumption on the optimal stage costs. Another formulation that solves a centralized MPC problem in distributed fashion can be found in [15] but no stability guarantees are given. In the DMPC literature some formulations do not solve a centralized problem but local optimization problems that take neighboring interaction into account, [4], [18], [10]. In [4], [18] stability (and robustness in the latter case) is guaranteed by letting the subsystems solve local optimization problems sequentially and pass the local solutions downstream to be used in the remaining local optimizations. In [10] stability is shown by setting explicit stabilizing constraints in the optimization. In the case of output feedback, there are quite few contributions in the DMPC literature. One exception is [19] in which nominal stability is proven using a decentralized estimator and local optimizations with full model data.

In this paper we extend the DMPC formulation presented in [8] to guarantee robustness to small disturbances using a constraint tightening approach and the inherent robustness of linear MPC. In [8] stability is shown without the use of a terminal constraint set which in many applications increases the region of attraction since there are no constraints on the end point. Using ideas from [12] we also propose an output feedback DMPC controller that is shown to be robustly stable and robustly feasible for small disturbances. Stability is shown by containing the estimation error within a positively robust invariant set and view the estimation error as a (bounded) disturbance. The inherent robustness of linear MPC is then used to show robust stability. To cope with the output feedback case, we restrict our treatment to systems with input couplings only since this allows for decentralized observer design. Such system descriptions arise, for instance, when flow between subsystems is controlled. The flow might be power in an electric network [1], water in hydro power valley [16] or intermediate products in a supply chain [5].

The paper is organized as follows. In Section II we formulate the problem and present useful results from [8]. In Section III we show robust stability and robust feasibility in the state feedback case. These results are used in Section IV to show robust stability and feasibility in the output feedback case. A numerical example is provided in Section V and the paper is concluded in Section VI.

II. Setup and preliminaries

We consider linear dynamical systems where each subsystem $i \in \{1, \ldots, M\}$ is described by

$$x_{t+1}^i = A_{ii}x_t^i + \sum_{j\in\mathcal{N}_i} B_{ij}u_t^j + w_t^i \qquad x_0^i = \bar{x}^i$$
$$y_t^i = C_i x_t^i + \xi_t^i$$

where $x_t^i \in \mathbb{R}^{n_i}$, $u_t^i \in \mathbb{R}^{m_i}$, $w_t^i \in \mathbb{R}^{n_i}$, $y_t^i \in \mathbb{R}^{p_i}$, $\xi_t^i \in \mathbb{R}^{p_i}$, and $\mathcal{N}_i$ is the neighboring interaction defined by

$$\mathcal{N}_i = \{j \in \{1, \ldots, M\} \mid B_{ij} \neq 0\}.$$

We assume that the system has some sparsity structure, i.e., that some $B_{ij} = 0$. We introduce the global variables $x = [(x^1)^T, \ldots, (x^M)^T]^T$, $u = [(u^1)^T, \ldots, (u^M)^T]^T$, $w = [(w^1)^T, \ldots, (w^M)^T]^T$, $y = [(y^1)^T, \ldots, (y^M)^T]^T$ and $\xi = [(\xi^1)^T, \ldots, (\xi^M)^T]^T$ where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $\xi \in \mathbb{R}^p$. This gives the following global system

$$x_{t+1} = Ax_t + Bu_t + w_t \qquad x_0 = \bar{x} \tag{1}$$
$$y_t = Cx_t + \xi_t \tag{2}$$

where the matrices A and C are block-diagonal and B is sparse. We assume hereafter that the pair (A, B) is stabilizable and the pair (A, C) is detectable. The local control and state variables as well as the disturbances are constrained, i.e., $u^i \in \mathcal{U}_i$, $x^i \in \mathcal{X}_i$, $w^i \in \mathcal{W}_i$ and $\xi^i \in \Xi_i$ where

$$\mathcal{X}_i = \{x^i \in \mathbb{R}^{n_i} \mid F_i^x x^i \leq g_i^x\},$$
$$\mathcal{U}_i = \{u^i \in \mathbb{R}^{m_i} \mid F_i^u u^i \leq g_i^u\},$$
$$\mathcal{W}_i = \{w^i \in \mathbb{R}^{n_i} \mid F_i^w w^i \leq g_i^w\},$$
$$\Xi_i = \{\xi^i \in \mathbb{R}^{p_i} \mid F_i^\xi \xi^i \leq g_i^\xi\}$$

where $F_i^x \in \mathbb{R}^{n_{f_{x^i}} \times n_i}$, $g_i^x \in \mathbb{R}^{n_{f_{x^i}}}$, $F_i^u \in \mathbb{R}^{n_{f_{u^i}} \times m_i}$, $g_i^u \in \mathbb{R}^{n_{f_{u^i}}}$, $F_i^w \in \mathbb{R}^{n_{f_{w^i}} \times n_i}$, $g_i^w \in \mathbb{R}^{n_{f_{w^i}}}$, $F_i^\xi \in \mathbb{R}^{n_{f_{\xi^i}} \times p_i}$ and $g_i^\xi \in \mathbb{R}^{n_{f_{\xi^i}}}$. We denote the total number of inequalities in $\mathcal{X}_i$ and $\mathcal{U}_i$ for all $i = 1, \ldots, M$ by q, i.e., $q = \sum_i (n_{f_{x^i}} + n_{f_{u^i}})$. The global constraint sets $\mathcal{X}, \mathcal{U}, \mathcal{W}$ and Ξ are defined as the set product of their respective local constraint sets. By introducing the predicted state and control vectors

$$\mathbf{z} = [z_0^T, \ldots, z_{N-1}^T]^T \qquad \mathbf{v} = [v_0^T, \ldots, v_{N-1}^T]^T \tag{3}$$

we formulate the following optimization problem which was used in the DMPC formulation in [8]

$$\begin{aligned} V_N(x) := \min_{\mathbf{z},\mathbf{v}} \quad & J_N(\mathbf{z}, \mathbf{v}) \\ \text{s.t.} \quad & z_\tau \in \mathcal{X}, && \tau = 0, \ldots, N-1, \\ & v_\tau \in \mathcal{U}, && \tau = 0, \ldots, N-1, \\ & z_{\tau+1} = Az_\tau + Bv_\tau, && \tau = 0, \ldots, N-2, \\ & z_0 = x. \end{aligned} \tag{4}$$

We denote the optimal state and control at time step τ for (4) by $z_\tau^*(x)$ and $v_\tau^*(x)$ respectively. The cost in (4) is assumed quadratic and separable

$$J_N(\mathbf{z}, \mathbf{v}) := \sum_{\tau=0}^{N-1} \ell(z_\tau, v_\tau) = \sum_{\tau=0}^{N-1}\sum_{i=1}^{M} \ell_i(z_\tau^i, v_\tau^i)$$
$$= \sum_{\tau=0}^{N-1}\sum_{i=1}^{M} \left(\frac{1}{2}(z_\tau^i)^T Q_i z_\tau^i + \frac{1}{2}(v_\tau^i)^T R_i v_\tau^i\right)$$

where $Q_i \succ 0$ and $R_i \succ 0$. Problem (4) can be solved efficiently in distributed fashion using the method developed in [7] which was also used in [8]. A short description of the optimization algorithm is given below. By introducing the vector $\boldsymbol{\chi} = [\mathbf{z}^T, \mathbf{v}^T]^T$ the optimization problem (4) can more compactly be written as

$$\begin{aligned} V_N(\bar{x}) := \min_{\boldsymbol{\chi}} \quad & \tfrac{1}{2}\boldsymbol{\chi}^T H \boldsymbol{\chi} \\ \text{s.t.} \quad & \mathbf{A}\boldsymbol{\chi} = \mathbf{b}x \\ & \mathbf{F}\boldsymbol{\chi} \leq \mathbf{g} \end{aligned}$$

where H and $\mathbf{F}$ are block-diagonal and $\mathbf{A}$ has the same structure as B in (1). We introduce dual variables $\boldsymbol{\mu} \in \mathbb{R}_{\geq 0}^{Nq}$ for the inequality constraints and $\boldsymbol{\lambda} \in \mathbb{R}^{n(N-1)}$ for the equality constraints. As shown in [7] the dual problem can be written as

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu}\geq 0} -\frac{1}{2}(\mathbf{A}^T\boldsymbol{\lambda} + \mathbf{F}^T\boldsymbol{\mu})^T H^{-1}(\mathbf{A}^T\boldsymbol{\lambda} + \mathbf{F}^T\boldsymbol{\mu}) - \boldsymbol{\lambda}^T\mathbf{b}x - \boldsymbol{\mu}^T\mathbf{g}. \tag{5}$$

The dual function was in [7] shown to have Lipschitz continuous gradient with Lipschitz constant $L = \|[\mathbf{A}^T \ \mathbf{F}^T]^T H^{-1}[\mathbf{A}^T \ \mathbf{F}^T]\|$ and can hence be maximized using accelerated gradient methods. The algorithm from [7] is presented here

$$\boldsymbol{\chi}^k = -H^{-1}\left(\mathbf{F}^T\boldsymbol{\mu}^k + \mathbf{A}^T\boldsymbol{\lambda}^k\right) \tag{6}$$
$$\bar{\boldsymbol{\chi}}^k = \boldsymbol{\chi}^k + \frac{k-1}{k+2}(\boldsymbol{\chi}^k - \boldsymbol{\chi}^{k-1}) \tag{7}$$
$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \frac{k-1}{k+2}(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}) + \frac{1}{L}\left(\mathbf{A}\bar{\boldsymbol{\chi}}^k - \mathbf{b}x\right) \tag{8}$$
$$\boldsymbol{\mu}^{k+1} = \max\left[0, \boldsymbol{\mu}^k + \frac{k-1}{k+2}(\boldsymbol{\mu}^k - \boldsymbol{\mu}^{k-1}) + \frac{1}{L}\left(\mathbf{F}\bar{\boldsymbol{\chi}}^k - \mathbf{g}\right)\right] \tag{9}$$

where k denotes the iteration number. Due to the structure of the matrices $\mathbf{A}, \mathbf{F}$ and H the algorithm can be implemented in distributed fashion where communication between subsystems i and j takes place if $j \in \mathcal{N}_i$ or $i \in \mathcal{N}_j$, see [7] fordetails. Further results from [7] shows that the algorithm converges as $O(\frac{1}{k^2})$ in dual function value. This is a significant enhancement compared to if the classical gradient method was used which converges as $O(\frac{1}{k})$.

In [8] feasibility, stability and performance of the closed loop system when solving (4), which has neither terminal cost nor terminal constraints, using (6)-(9) was established. Since (6)-(9) gives a primal feasible solution only in the limit of iterations, an adaptive constraint tightening approach was used to ensure feasibility, stability, and performance with finite number of algorithm iterations. However, in this paper we state all results as if the optimal solution to (6)-(9) is found in each iteration. The generalization to allow for early termination using the stopping condition in [8] is straightforward but requires quite some notation to be introduced. We introduce

$$\mathbb{X}_N := \{x \in \mathbb{R}^n \mid V_N(x) < \infty \text{ and } Az_{N-1}^*(x) \in \mathcal{X}\}.$$

We also define the infinite horizon steerable set

$$\mathbb{X}_\infty := \{x \in \mathbb{R}^n \mid V_\infty(x) < \infty\}$$

and $\ell^*(x) := \frac{1}{2}x^T Q x$ and the following definition:

Definition 1: The constant Φ_N is the smallest constant such that the optimal solution $\{z^*_\tau(x)\}_{\tau=0}^{N-1}$, $\{v^*_\tau(x)\}_{\tau=0}^{N-1}$ to (4) for every $x \in \mathbb{X}_N$ satisfies

$$\ell^*(z^*_{N-1}(x)) \le \Phi_N \ell(x, v^*_0(x)) \tag{10}$$

for the chosen control horizon N.

We introduce the optimal feedback control law $\nu_N(x) := v^*_0(x)$ and define the nominal and actual next states

$$\begin{aligned}\bar{x}_{t+1} &:= Ax_t + B\nu_N(x_t)\\ x_{t+1} &:= Ax_t + B\nu_N(x_t) + w_t\end{aligned}$$

where $w_t \in \mathcal{W}$. We define $\kappa = \|Q^{-1/2}A^T Q A Q^{-1/2}\|_2$ and state the following result from [8, Corollary 1].

Theorem 1: Suppose that $\alpha \le 1 - \kappa\Phi_N$. Then

$$V_N(x) \ge V_N(Ax + B\nu_N(x)) + \alpha\ell(x, \nu_N(x))$$

holds for every $x \in \mathbb{X}_N$.

Throughout the remainder of the paper we assume that $\alpha > 0$ and N are chosen in accordance with Theorem 1.

Assumption 1: We assume that the disturbance sets $\mathcal{W}, \varXi$ are bounded and that $0 \in \mathrm{int}\mathcal{W}$, $0 \in \mathrm{int}\varXi$. Further we assume that $\mathcal{B}^{n_i}_\infty(r_x) \subset \mathcal{X}_i$, $\mathcal{B}^{m_i}_\infty(r_u) \subset \mathcal{U}_i$ for some $r_x, r_u > 0$ where $\mathcal{B}^n_\infty(r)$ is defined in (11).

A. Notation

The norm ball is defined as

$$\mathcal{B}^n_l(r) := \{x \in \mathbb{R}^n \mid \|x\|_l \le r\}. \tag{11}$$

The $\oplus$ denotes the Minkowski sum defined by $\mathcal{X}_1 \oplus \mathcal{X}_2 \triangleq \{x_1 + x_2 \mid x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\}$ and $\ominus$ denotes the Pontryagin difference defined by

$$\mathcal{X}_1 \ominus \mathcal{X}_2 \triangleq \{x \mid \{x\} \oplus \mathcal{X}_2 \subseteq \mathcal{X}_1\}. \tag{12}$$

Finally $h_{\mathcal{X}}(\theta)$ is the support function which is defined as $h_{\mathcal{X}}(\theta) \triangleq \sup_{x\in\mathcal{X}} \theta^T x$.

Remark 1: For polytopic sets $\mathcal{X}_1 = \{x \in \mathbb{R}^n \mid X_1 x \le y_1\}$, $\mathcal{X}_2 = \{x \in \mathbb{R}^n \mid X_2 x \le y_2\}$ we have from [11, Theorem 2.3] that

$$\mathcal{X}_1 \ominus \mathcal{X}_2 = \{x \in \mathbb{R}^n \mid [X_1]_j x \le [y_1]_j - h_{\mathcal{X}_2}([X_1]_j^T),\ j = 1, \ldots, p\}$$

where X_1 has p rows, $[X_1]_j$ is the j:th row of X_1 and $[y_1]_j$ is the j:th element of y_1. Thus, $\mathcal{X}_1 \ominus \mathcal{X}_2$ and $\mathcal{X}_1$ can be described using the same number of linear inequalities.

III. Robust state feedback DMPC

In this section we consider the state feedback problem, i.e., with $C = I$ and $\xi = 0$ in (2). We will see that by tightening the constraints in the optimization problem we can guarantee robust stability and robust feasibility. We start by investigating robust feasibility.

A. One-step robust feasibility

To guarantee that the system is one-step robustly feasible, a constraint tightening approach is used. We introduce the sets $\mathcal{X}_i \ominus \mathcal{W}_i$ for $i = 1, \ldots, M$ which can be computed as in Remark 1. Since the number of constraints that describes $\mathcal{X}_i \ominus \mathcal{W}_i$ is the same as the number of constraints that describes $\mathcal{X}_i$, these tightened constraint sets can be used in the optimization without increasing the complexity. Defining the corresponding global constraint set $\mathcal{X} \ominus \mathcal{W}$ as the set product of the local sets, we get the following optimization problem with tightened constraints

$$\begin{aligned} V_N(x) := \min_{\mathbf{z},\mathbf{v}} \quad & J_N(\mathbf{z}, \mathbf{v}) \\ \text{s.t.} \quad & z_\tau \in \mathcal{X} \ominus \mathcal{W}, && \tau = 0, \ldots, N-1, \\ & v_\tau \in \mathcal{U}, && \tau = 0, \ldots, N-1, \\ & z_{\tau+1} = Az_\tau + Bv_\tau, && \tau = 0, \ldots, N-2, \\ & z(0) = x. \end{aligned} \tag{13}$$

The state constraint set is changed in (13) compared to in (4). Thus, we get a different control law ν_N, infinite horizon steerable set $\mathbb{X}_\infty$, set $\mathbb{X}_N$, and value function V_N. To avoid introducing new notation we use the same notation but the quantities are in this section based on optimization problem (13) instead of (4). The following proposition shows one-step robust feasibility.

Proposition 1: For any $x_t \in \mathbb{X}_N$ we have that $x_{t+1} \in \mathcal{X}$ for any disturbance $w_t \in \mathcal{W}$.

Proof. From the problem formulation we have that $\bar{x}_{t+1} \in \mathcal{X} \ominus \mathcal{W}$. From [11, Theorem 2.1] we know that $(\mathcal{X} \ominus \mathcal{W}) \oplus \mathcal{W} \subseteq \mathcal{X}$. Further, $x_{t+1} = \bar{x}_{t+1} + w_t \in (\mathcal{X} \ominus \mathcal{W}) \oplus \mathcal{W} \subseteq \mathcal{X}$. This concludes the proof. □

This shows that if the optimization problem is feasible, we get one-step robust feasibility.

B. Robust stability

For systems with linear dynamics, quadratic cost and polytopic constraints we know that the value function is continuous [9], [2]. Thus, for every $x \in \mathbb{X}_N \ominus \mathcal{W}$ we have for some finite $\beta_w \ge 0$ that

$$\max_{w\in\mathcal{W}} V_N(x + w) - V_N(x) \le \beta_w \tag{14}$$

since $x + w \in \mathbb{X}_N$ for any $x \in \mathbb{X}_N \ominus \mathcal{W}$ and $w \in \mathcal{W}$. This observation is used to prove inherent robustness of the closed loop system to small disturbance sets $\mathcal{W}$. To show robust stability we need to introduce some sets. The first is the following ellipsoid

$$\mathcal{E}(\gamma) := \{x \in \mathbb{R}^n \mid (\alpha - \epsilon)\ell^*(x) \le \gamma\} \tag{15}$$

where $\epsilon > 0$ is small and $\alpha > \epsilon$ is from Theorem 1. The second is the value function level sets

$$\Omega(c) := \{x \in \mathbb{R}^n \mid V_N(x) \le c\}.$$

We also introduce the following recursive definition of the maximal positively robust invariant set

$$\mathbb{X}_{\mathrm{rf}} = \{x \in \mathbb{X}_N \mid \{Ax + B\nu_N(x)\} \oplus \mathcal{W} \subseteq \mathbb{X}_{\mathrm{rf}}\}.$$

Before we state the theorem about asymptotic convergence, we need the following assumption.

Assumption 2: We assume that the disturbance set $\mathcal{W}$ is small enough to guarantee $\Omega(\delta) \subset \mathbb{X}_{\mathrm{rf}}$ where $\delta = 2\max_{x\in\mathcal{E}(\beta_w)} V_N(x)$.

Theorem 2: Suppose that Assumption 2 holds. Then for any initial condition $x_0 \in \mathbb{X}_{\mathrm{rf}}$, the closed loop system is asymptotically converging to $\Omega(\delta)$, where $\delta = 2\max_{x\in\mathcal{E}(\beta_w)} V_N(x)$. Further, $x_t \in \mathcal{X}$ for all $t \geq 0$.

Proof. For any $x_t \in \mathbb{X}_{\mathrm{rf}} \backslash \mathcal{E}(\beta_w)$ we have

$$\begin{aligned} V_N(x_t) &\geq V_N(\bar{x}_{t+1}) + \alpha\ell(x_t, \nu_N(x_t)) + \\ &\quad + \max_{w\in\mathcal{W}} V_N(\bar{x}_{t+1} + w_t) - \max_{w\in\mathcal{W}} V_N(\bar{x}_{t+1} + w_t) \\ &\geq \max_{w\in\mathcal{W}} V_N(\bar{x}_{t+1} + w_t) + \alpha\ell^*(x_t) - \beta_w \\ &\geq V_N(x_{t+1}) + \epsilon\ell^*(x_t) \geq V_N(x_{t+1}) + \frac{\epsilon\beta_w}{\alpha - \epsilon} \end{aligned}$$

where the first inequality comes from Theorem 1 since $x_t \in \mathbb{X}_{\mathrm{rf}} \backslash \mathcal{E}(\beta_w) \subseteq \mathbb{X}_N$. The second inequality is by definition of ℓ^* and from (14) since by definition of $\mathbb{X}_{\mathrm{rf}}$ and of $\ominus$ we have $\bar{x}_{t+1} \in \mathbb{X}_{\mathrm{rf}} \ominus \mathcal{W} \subseteq \mathbb{X}_N \ominus \mathcal{W}$. The third and fourth inequalities are from (15) since $x_t \notin \mathcal{E}(\beta_w)$. By definition of δ we have $\mathcal{E}(\beta_w) \subseteq \Omega(\delta/2)$ which implies $\mathbb{X}_{\mathrm{rf}} \backslash \Omega(\delta/2) \subseteq \mathbb{X}_{\mathrm{rf}} \backslash \mathcal{E}(\beta_w)$. This implies that for any $x_t \in \mathbb{X}_{\mathrm{rf}} \backslash \Omega(\delta/2)$ we have

$$V_N(x_t) \geq V_N(x_{t+1}) + \frac{\epsilon\beta_w}{\alpha - \epsilon}. \tag{16}$$

By definition of $\mathbb{X}_{\mathrm{rf}}$ we have $x_{t+1} \in \mathbb{X}_{\mathrm{rf}}$ which implies that the preceeding argument can be applied recursively. Thus, for any initial state $x_0 \in \mathbb{X}_{\mathrm{rf}} \backslash \Omega(\delta/2)$ there is a finite time $t = t_0$ such that $x_{t_0} \in \Omega(\delta/2)$. Note that if $x_0 \in \Omega(\delta/2)$ we get $t_0 = 0$.

The system state can leave $\Omega(\delta/2)$ ones entered. However, the departure from this set is bounded. We have that

$$\frac{\delta}{2} = \max_{x\in\Omega(\delta_2)} V_N(x) \geq \max_{x\in\Omega(\delta_2)} \ell^*(x) \geq \max_{x\in\mathcal{E}(\beta_w)} \ell^*(x) = \frac{\beta_w}{\alpha - \epsilon}.$$

This gives that for every $x_t \in \Omega(\delta/2)$ we have

$$\begin{aligned} \max_{w\in\mathcal{W}} V_N(\bar{x}_{t+1} + w_t) &\leq V_N(x_t) - \alpha\ell(x_t, \nu_N(x_t)) + \beta_w \\ &\leq V_N(x_t) + \beta_w \\ &\leq \frac{\delta}{2} + \beta_w \leq \frac{\delta}{2}(1 + \alpha - \epsilon) \leq \delta. \end{aligned}$$

Thus, for $x_t \in \Omega(\delta/2)$ we have $x_{t+1} \in \Omega(\delta)$ for any $w \in \mathcal{W}$. Since by Assumption 2 we have $\Omega(\delta) \subset \mathbb{X}_{\mathrm{rf}}$ get from (16) that system never leaves $\Omega(\delta)$.

To show that $x_t \in \mathcal{X}$ for all $t \geq 0$ we note due to the definition of $\mathbb{X}_{\mathrm{rf}}$ that $\bar{x}_{t+1} \in \mathbb{X}_{\mathrm{rf}} \ominus \mathcal{W}$ for any $t \geq 1$. This implies that $x_{t+1} = \bar{x}_{t+1} + w_t \in (\mathbb{X}_{\mathrm{rf}} \ominus \mathcal{W}) \oplus \mathcal{W} \subseteq \mathbb{X}_{\mathrm{rf}} \subseteq \mathcal{X}$ for any $t \geq 1$.

This completes the proof. □

In the following section we will see that the result presented in this section can be used to prove robust stability and robust feasibility for output feedback DMPC.

IV. Output feedback DMPC

We will use the result presented in the previous section to prove feasibility and stability properties in the output feedback setting. We start by designing the observer.

A. Observer Design

A crucial part for keeping the resulting output feedback controller simple is that the observer design can be performed in decentralized fashion. With the assumed structure on the dynamics, i.e., block-diagonal A-matrix, we can design local observers for each subsystem. In each subsystem the following observer is used

$$\hat{x}^i_{t+1} = A_{ii}\hat{x}^i_t + \sum_{j\in\mathcal{N}_i} \left(B_{ij}u^j_t\right) + K_i(y^i_t - C_i\hat{x}^i_t).$$

The information, besides the local information, needed to update the local estimates are the control action from neighboring nodes. This information is available in node i from the optimization algorithm communications. The local observers together form the following global observer

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + K(y_t - C\hat{x}_t) \tag{17}$$

where $K = \mathrm{blkdiag}(K_1, \ldots, K_M)$. The error dynamics for the observer is purely local. We introduce the local error variables as $\widetilde{x}^i = x^i - \hat{x}^i$ and get the following local error dynamics

$$\begin{aligned} \widetilde{x}^i_{t+1} &= A_{ii}x^i_t + \sum_{j\in\mathcal{N}_i} \left(B_{ij}u^j_t\right) + w^i_t - \\ &\quad - A_{ii}\hat{x}^i_t - \sum_{j\in\mathcal{N}_i} \left(B_{ij}u^j_t\right) - K_i(y^i_t - C_i\hat{x}^i_t) \\ &= (A_{ii} - K_iC_i)\widetilde{x}^i_t - K_i\xi^i_t + w^i_t. \end{aligned}$$

This shows that the poles of the observer dynamics can be placed arbitrarily using a block-diagonal observer gain K. For given K_i such that $\rho(A_{ii} - K_iC_i) < 1$ there exists a robust invariant set for the estimation error [11]. In [17] it was shown how an invariant outer approximation of the minimal robust invariant set can be computed. The minimal robust invariant set is (cf. [17])

$$\mathcal{R}_i = \bigoplus_{j=0}^{\infty} \mathcal{F}^j_i$$

where $\mathcal{F}^j_i := (A_{ii} - K_iC_i)^j[-K_i\Xi_i \oplus \mathcal{W}_i]$. In the approximation only a finite number of terms in the Minkowski sum is used and the resulting set sum is scaled to guarantee a certain accuracy of the approximation. The approximation is

$$\mathcal{R}^{\epsilon_e}_i = \frac{1}{1 - \kappa_i} \bigoplus_{j=0}^{s_i} \mathcal{F}^j_i$$

where s_i and κ_i can, for given accuracy ϵ_e, be computed without performing the Minkowski summation (cf. [17]). The approximation is also robust invariant and satisfies (cf. [17])

$$\mathcal{R}_i \subseteq \mathcal{R}^{\epsilon_e}_i \subseteq \mathcal{R}_i \oplus \mathcal{B}^{n_i}_\infty(\epsilon_e).$$

From the definition of a robust invariant set we get that if $\widetilde{x}_0^i \in \mathcal{R}_i^{\epsilon_e}$ we have $\widetilde{x}_t^i \in \mathcal{R}_i^{\epsilon_e}$ for all $t \geq 0$ and any disturbance sequences $\{\xi_t^i\}_{t=0}^{\infty}$, $\{w_t^i\}_{t=0}^{\infty}$. We define the global robust invariant set as $\mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_M$ and the approximation $\mathcal{R}^{\epsilon_e}$ accordingly. We get

$$\mathcal{R} \subseteq \mathcal{R}^{\epsilon_e} \subseteq \mathcal{R} \oplus \mathcal{B}_\infty^n(\epsilon_e)$$

since $\mathcal{B}_\infty^n(\epsilon_e) = \mathcal{B}_\infty^{n_1}(\epsilon_e) \times \ldots \times \mathcal{B}_\infty^{n_M}(\epsilon_e)$.

B. One-step robust feasibility

The feedback in the output feedback case is based on the estimated current state $\hat{x}_t$. The objective of this section is to show how the original constraints need to be tightened to guarantee feasibility of the next state x_{t+1} and the estimated next state $\hat{x}_{t+1}$ for any disturbances $w \in \mathcal{W}$, $\xi \in \Xi$. We rewrite the observer dynamics (17) as

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + \hat{w}_t, \qquad \hat{w}_t = K(C\widetilde{x}_t + \xi_t) \tag{18}$$

and introduce the following set $\mathcal{W}_o = KC\mathcal{R}^{\epsilon_e} \oplus K\Xi$ and the corresponding local sets $\mathcal{W}_{o,i} = K_iC_i\mathcal{R}_i^{\epsilon_e} \oplus K_i\Xi_i$. We will see that the following optimization problem gives one-step robust feasibility when the initial condition is the estimated state:

$$\begin{aligned} V_N(\hat{x}) := \min_{\mathbf{z},\mathbf{v}} \quad & J_N(\mathbf{z},\mathbf{v}) \\ \text{s.t.} \quad & z_\tau \in \mathcal{X} \ominus \mathcal{W}_o \ominus \mathcal{R}^{\epsilon_e}, && \tau = 0,\ldots,N-1, \\ & v_\tau \in \mathcal{U}, && \tau = 0,\ldots,N-1, \\ & z_{\tau+1} = Az_\tau + Bv_\tau, && \tau = 0,\ldots,N-2, \\ & z_0 = \hat{x}. \end{aligned} \tag{19}$$

Remark 2: The tightened state constraint set $\mathcal{X} \ominus \mathcal{W}_o \ominus \mathcal{R}^{\epsilon_e}$ is the product of the corresponding tightened local constraint sets $\mathcal{X}_i \ominus \mathcal{W}_{o,i} \ominus \mathcal{R}_i^{\epsilon_e}$ which can be computed efficiently by noting that

$$\begin{aligned} &\mathcal{X}_i^\epsilon \ominus \mathcal{W}_{o,i} \ominus \mathcal{R}_i^{\epsilon_e} = \mathcal{X}_i \ominus K_iC_i\mathcal{R}_i^{\epsilon_e} \ominus K_i\Xi_i \ominus \mathcal{R}_i^{\epsilon_e} \\ &= \mathcal{X}_i^\epsilon \left(\bigominus_{j=0}^{s_i} (\frac{K_iC_i}{1-\kappa_i}\mathcal{F}_i^j) \ominus K_i\Xi_i \bigominus_{j=0}^{s_i} (\frac{1}{1-\kappa_i}\mathcal{F}_i^j) \right) \\ &= (X_i^\epsilon \ominus \frac{K_iC_i}{1-\kappa_i}\mathcal{F}_i^0) \left(\bigominus_{j=1}^{s_i} (\frac{K_iC_i}{1-\kappa_i}\mathcal{F}_i^j) \ominus K_i\Xi_i \ominus \mathcal{R}_i^{\epsilon_e} \right) \end{aligned}$$

where [11, Theorem 2.1] is used in all steps. This implies that the local tightened constraint set can be computed by taking the Pontryagin difference $\ominus$ recursively set by set. The number of inequalities that describes the final tightened constraint set is the same as in $\mathcal{X}_i$ due to Remark 1. This way, an explicit description of $\mathcal{R}_i^{\epsilon_e}$, which can be very expensive to compute, is avoided.

The new optimization problem with tightened constraints gives a new feedback control law ν_N, infinite horizon steerable set $\mathbb{X}_\infty$, set $\mathbb{X}_N$ and value function V_N. The notation is kept from previous sections, but the respective definitions refer in this section to optimization problem (19). Also, the definition of the recursively feasible set is different, we define

$$\mathbb{X}_{\text{rf}} = \{\hat{x} \in \mathbb{X}_N \mid (\{A\hat{x} + B\nu_N(\hat{x})\} \oplus \mathcal{R}^{\epsilon_e}) \oplus \mathcal{W}_o \subseteq \mathbb{X}^{\text{rf}}\}.$$

We also define the one-step nominal prediction

$$\bar{x}_{t+1} := A\hat{x}_t + B\nu_N(\hat{x}_t).$$

The following proposition shows that when using optimization problem (19) one-step robust feasibility in plant state x and estimated state $\hat{x}$ is achieved regardless of disturbances $w \in \mathcal{W}$, $\xi \in \Xi$.

Proposition 2: Suppose that $\widetilde{x}_t \in \mathcal{R}^{\epsilon_e}$ and $\hat{x}_t \in \mathbb{X}_N$. Then $\hat{x}_{t+1} \in \mathcal{X} \ominus \mathcal{R}^{\epsilon_e}$ and $x_{t+1} \in \mathcal{X}$.

Proof. From the problem formulation we have that $\bar{x}_{t+1} \in \mathcal{X} \ominus \mathcal{R}^{\epsilon_e} \ominus \mathcal{W}_o$. Further $\hat{x}_{t+1} = \bar{x}_{t+1} + \hat{w}_t \in (\mathcal{X} \ominus \mathcal{R}^{\epsilon_e} \ominus \mathcal{W}_o) \oplus \mathcal{W}_o \subseteq \mathcal{X} \ominus \mathcal{R}^{\epsilon_e}$. Since $\widetilde{x}_t \in \mathcal{R}^{\epsilon_e}$ we have $\widetilde{x}_{t+1} \in \mathcal{R}^{\epsilon_e}$ and $x_{t+1} = \hat{x}_{t+1} + \widetilde{x}_{t+1} \in \mathcal{X} \ominus \mathcal{R}^{\epsilon_e} \oplus \mathcal{R}^{\epsilon_e} \subseteq \mathcal{X}$. This concludes the proof. □

C. Robust stability

The estimation is affected by additive noise $\hat{w}_t$ which satisfies $\hat{w}_t \in \mathcal{W}_o$ for all $t \geq 0$ if the estimation error $\widetilde{x}_t \in \mathcal{R}^{\epsilon_e}$ for all $t \geq 0$. From the discussion in Section III we conclude that for every $x \in \mathbb{X}_N \ominus \mathcal{W}_o$ we have with finite $\beta_{w_o} \geq 0$ that

$$\max_{\hat{w} \in \mathcal{W}_o} V_N(x + \hat{w}) - V_N(x) \leq \beta_{w_o}.$$

In the following theorem we show that the estimated state $\hat{x}_t$ and plant state x_t converges to robust invariant sets. Before we state the theorem, the following assumption is needed.

Assumption 3: We assume that the disturbance sets $\mathcal{W}$ and Ξ are small enough to guarantee $\Omega(\delta) \subset \mathbb{X}_{\text{rf}}$ where $\delta = 2\max_{x \in \mathcal{E}(\beta_{w_0})} V_N(x)$.

Theorem 3: Suppose that Assumption 3 holds and that $\widetilde{x}_0 = x_0 - \hat{x}_0 \in \mathcal{R}^\epsilon$. Then for any $\hat{x}_0 \in \mathbb{X}_{\text{rf}}$ the state estimation $\hat{x}_t$ converges to $\Omega(\delta)$ where $\delta = 2\max_{x \in \mathcal{E}(\beta_{w_o})} V_N(x)$ and the plant state x_t converges to $\Omega(\delta) \oplus \mathcal{R}^\epsilon$. Further $x_t \in \mathcal{X}$ for all $t \geq 1$.

Proof. Since $\widetilde{x}_0 \in \mathcal{R}^\epsilon$ we have $\widetilde{x}_t \in \mathcal{R}^\epsilon$ for all $t \geq 0$. This implies that the disturbance to the estimated state (18) satisfies $\hat{w}_t \in \mathcal{W}_o$ for all $t \geq 0$. Convergence of the estimated state $\hat{x}_t$ to $\Omega(\delta)$ is then given by Theorem 2 since the situation for $\hat{x}_t$ is analogous to the situation for x_t in Theorem 2. Further, Theorem 2 also gives together with the definition of $\mathbb{X}_{\text{rf}}$ that $\hat{x}_t \in \mathbb{X}_{\text{rf}} \ominus \mathcal{R}^{\epsilon_e}$ for all $t \geq 1$.

Convergence of the plant state $x_t = \hat{x}_t + \widetilde{x}_t$ to $\Omega(\delta) \oplus \mathcal{R}^{\epsilon_e}$ follows directly from the estimated state $\hat{x}_t$ convergence to $\Omega(\delta)$ and since $\widetilde{x}_t \in \mathcal{R}^{\epsilon_e}$ for all $t \geq 0$. That $x_t \in \mathcal{X}$ for all $t \geq 1$ follows directly from $x_t = \hat{x}_t + \widetilde{x}_t \in (\mathbb{X}_{\text{rf}} \ominus \mathcal{R}^{\epsilon_e}) \oplus \mathcal{R}^{\epsilon_e} \subseteq \mathbb{X}_{\text{rf}} \subseteq \mathcal{X}$.

This concludes the proof. □

V. Numerical Example

We evaluate the efficiency of the proposed output feedback controller by applying it to a randomly generated system. The system is composed of six subsystems with five states, one control signal, and one output each. The measurement and system noise are bounded and within the following sets

$$\Xi_i = \{\xi^i \in \mathbb{R} \mid |\xi^i| \leq 0.01\},$$

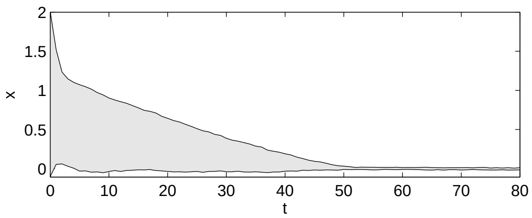

Fig. 1. Region within which all state trajectories are confined for $N = 15$ in the output feedback case. Guaranteed upper and lower bounds for the all state variables are $-0.11 \leq x \leq 2$.

$$\mathcal{W}_i = \{w^i \in \mathbb{R}^5 \mid \|w^i\|_\infty \leq 0.01\}$$

and the state and control constraint sets are

$$\begin{aligned}&\mathcal{U}_i = \{u^i \in \mathbb{R} \mid |u^i| \leq 0.1\},\\&\mathcal{X}_i = \{x^i \in \mathbb{R}^5 \mid -0.11 \leq [x^i]_j \leq 2, j = 1, \ldots, 5\}.\end{aligned}$$

The observer gain is chosen as Kalman gain computed using unit noise variances. The tightened constraint set $\mathcal{X}_i^\epsilon \ominus K_i C_i \mathcal{R}_i^{\epsilon_e} \ominus K_i \Xi_i \ominus \mathcal{R}_i^{\epsilon_e}$ is computed using accuracy $\epsilon_e = 0.0001$ in $\mathcal{R}_i^{\epsilon_e}$. The resulting set has upper bounds on all state variables in the range $[1.895, 1.959]$ and lower bounds on the state variables in the range $[-0.069, -0.005]$. The nominal next state must satisfy these constraints to ensure that the estimated and true states satisfy the original constraints defined by $\mathcal{X}_i$. State and control costs are chosen, $Q = I$, $R = I$.

Numerical simulations suggest that for $\alpha = 0.5$ we get $N = 15$ in Theorem 1 and for $\alpha = 0.2$ we get $N = 6$. In Figure 1 the largest and smallest state values for each time step are plotted. The initial state vector comes from a uniform distribution and is scaled such that the largest element in the vector equals the original upper bound, i.e., 2 and the smallest element in the vector equals the original lower bound, i.e., -0.11.

We also analyze the size of the region of attraction and compare it to standard MPC where the terminal set is chosen as the maximal positive invariant set for the LQ-feedback computed using $Q = I$, $R = I$ (see [6]). The system is initialized with 40000 different initial conditions and each element in the initial state vector is chosen from a uniform distribution in the interval $[-0.11\ 2]$, i.e., in the original constraint set. We have made two comparisons, the first is with $\alpha = 0.2$ which gives $N = 6$. Using $N = 6$ our controller managed to steer 98.9% of the initial conditions to the origin while respecting all constraints. The corresponding number in standard MPC with terminal constraint set and $N = 6$, was that 21.9% of the initial conditions were controlled to the origin. In the case for $\alpha = 0.5$ which gives $N = 15$ our controller managed to steer 98.9% of the initial conditions to the origin. For standard MPC with $N = 15$ the corresponding number was 52.8%. Note that the same set of initial conditions was used for all controllers. This shows that by not using a terminal constraint set, the region of attraction can be increased significantly while the computational burden is reduced.

VI. Conclusions

A robust distributed output feedback DMPC controller is proposed where the nominal behavior is optimized and the optimization problem has no terminal constraint set or terminal cost. Nominal stability for such DMPC formulations was proven in [8]. The results in [8] are in this paper extended to show robust stability in the state feedback case as well as the output feedback case. The provided numerical example also suggests that the lack of terminal constraint set can increase the region of attraction significantly.

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