Optimal movement in 2D for DOA estimation

Let say we have a given time interval available and can move at a maximum velocity of 1. We want to find the best trajectory to move a receiver antenna that optimizes the CRB performance of DOA estimation of an incoming planar radio wave, regardless of its incoming direction. Since the accuracy is determined by the moments of inertia this amounts to finding a unit length trajectory that maximizes the minimal moment of inertia of a string of unit mass density.

To simplify lets assume the movement is in 2D, $(x(t), y(t))$. Lets also assume (prove?) we have a symmetry axis in the movement along the y-axis so that it can be described by $(x(t), y(t),\ t \in [-1, 1]$ where $(x(-t), y(-t)) = (-x(t), y(t)), \forall t$. Also assume $x(0) = y(0) = 0$.

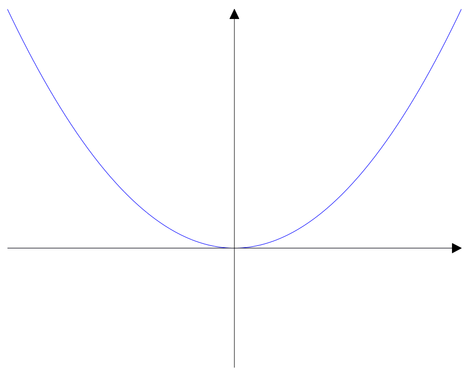

Figure 1: The trajectory is assumed symmetric as in the figure

Note that the moment of inertia matrix $\begin{pmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{pmatrix}$ is diagonal since $I_{xy} = 0$ because of the symmetry. To maximize $\min(I_{xx}, I_{yy})$ we will study $\max \gamma I_{xx} + I_{yy}$ for different $\gamma > 0$. The optimization problem can now be formulated as this optimal control problem

$$
\begin{aligned}
&\max\ \gamma I_{xx} + I_{yy} \quad \text{subject to} \\
&I_{xx} = \int_0^1 x^2(t)dt \\
&I_{yy} = \int_0^1 y^2(t)dt - \left(\int_0^1 y(t)dt\right)^2 \\
&\dot{x}^2(t) + \dot{y}^2(t) \leq 1
\end{aligned}
$$

which can be formulated as the optimal control problem

$$
\begin{aligned}
&\max \int_0^1 \gamma x_1^2(t) + x_2^2(t)dt - x_3^2(1) \quad \text{subject to} \\
&\dot{x}_1 = u_1 \\
&\dot{x}_2 = u_2 \\
&\dot{x}_3 = x_2 \\
&u_1^2 + u_2^2 \leq 1
\end{aligned}
$$

Probably we can also assume that $u_1 \geq 0$ and $u_2 \geq 0$.

Pontryagin's minimum principle now gives with adjoint variable $p(t)$, running cost $L := -\frac{1}{2}(\gamma x_1^2 + x_2^2)$ and final cost $K := \frac{1}{2}x_3^2(1)$

$$\begin{aligned} H &= p^T f + L = p_1 u_1 + p_2 u_2 + p_3 x_2 - \frac{1}{2}\gamma x_1^2 - \frac{1}{2}x_2^2 \\ \dot{p} &= -H_x \\ p(1) &= K_x \end{aligned}$$

which gives

$$\begin{aligned} \dot{x}_1 &= u_1, & x_1(0) &= 0 \\ \dot{x}_2 &= u_2, & x_2(0) &= 0 \\ \dot{x}_3 &= x_2, & x_3(0) &= 0 \\ \dot{p}_1 &= \gamma x_1, & p_1(1) &= 0 \\ \dot{p}_2 &= x_2 - p_3, & p_2(1) &= 0 \\ \dot{p}_3 &= 0, & p_3(1) &= x_3(1) \end{aligned}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^* = \operatorname{argmin}_u H = -\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Big/ \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|$$

Remark: These equations always have one (often false) solution with $x_2 = x_3 = p_2 = p_3 = 0$ giving $(x_1, x_2) = (t, 0)$, i.e. a straight line. This is probably optimal for large γ. One can also notice that since $p_2(t) = p_2(0) + x_3(t) - p_3 t$ it follows that for $t = 1$ that $p_2(1) = p_2(0) + x_3(1) - p_3(1)$, giving $p_2(0) = 0$.

With $\gamma = 0.20333$ we get $I_x \approx I_y \approx 0.07136$ and the solution is shown in the following figure. It is remarkable that the solution is extremly close to (equal to?) the curve

$$y = 4x^2 + 16x^4.$$

also shown in the same figure.

This total curve has length $= 2$. Rescaling to unit length gives a curve with $I_x = I_y = 0.01784$. This is 41 percent better than the result for a circle with unit length, which has $I_x = I_y = \frac{1}{8\pi^2} = 0.0127$, and 72 percent better than a square which has $I_x = I_y = \frac{4}{384} = 0.0104$. A wedge with opening angle 60 degrees (best) has $I_x = I_y = \sqrt{3}/128 = 0.0135$.

Code at `/home/bob/doktorander/mannesson/tests/doaoptim/` :

```
solinit = bvpinit(linspace(0,1,50),[0.3 0.4 0.2 -0.1 -0.1 0.3]);
options = bvpset('Stats','on','RelTol',1e-7);
global gamma; gamma = 0.20333;
sol = bvp4c(@BVP_ode, @BVP_bc, solinit, options);
t = sol.x;
xp = sol.y;
%%%%%%%%%%%%%%%%%%%
function dxpdt = BVP_ode(t,xp )
global gamma;
x1 = xp(1);
x2 = xp(2);
x3 = xp(3);
```

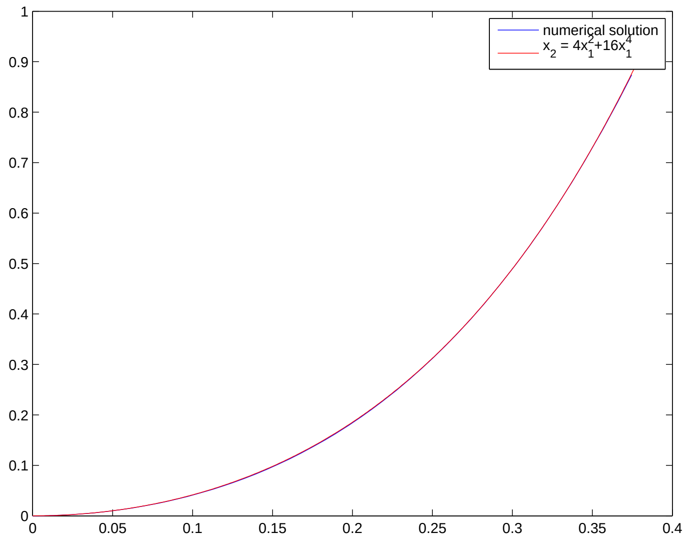

Figure 2: Numerical solution when $\gamma = 0.20333$. Only half the curve is plotted, the part with $x_1 < 0$ is symmetric. The curve is indistinguishable from a 4th order polynomial.

```
p1 = xp(4);
p2 = xp(5);
p3 = xp(6);
pnorm = norm([p1;p2]);
u1 = -p1/pnorm;
u2 = -p2/pnorm;
dxpdt = [u1; u2; x2; gamma*x1; x2-p3; 0];
%%%%%%%%%%%%%%%%%%%%%
function res = BVP_bc(xpinit,xpfinal )
res = [xpinit(1); xpinit(2); xpinit(3); xpfinal(4); xpfinal(5); xpfinal(6)-xpfinal(3)];
```

Open Questions: Can we prove the curve is symmetric? Is it a 4th order polynomial, or is it just very close? What does the optimal 3D movement look like? If one instead of maximizing the minimial moment of inertia, which focuses on the worst case direction, optimizes performance for a random direction of the incoming ray. Will it give the same solution? What if one must come back to the initial point, will it be a circle then? Probably one can use similar optimization problem but enforce $x_1(t_f) = 0$.

Returning to initial point

If we enforce that the solution returns to the initial point, a natural guess is that the optimal trajectory is a circle. With the same setup as before, i.e. assuming a symmetry axis along the y-axis the corresponding equations from Pontryagin's principle become

$$
\begin{aligned}
\dot{x}_1 &= u_1, & x_1(0) &= 0, x_1(1) = 0\\
\dot{x}_2 &= u_2, & x_2(0) &= 0\\
\dot{x}_3 &= x_2, & x_3(0) &= 0\\
\dot{p}_1 &= \gamma x_1, & p_1(1) &\text{ free}\\
\dot{p}_2 &= x_2 - p_3, & p_2(1) &= 0\\
\dot{p}_3 &= 0, & p_3(1) &= x_3(1)\\
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^* &= \operatorname{argmin}_u H = -\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Big/ \left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|
\end{aligned}
$$

It is now easy to verify that the the equations with $\gamma = 1$ are fulfilled by the semi-circle $x_1(t) = \frac{1}{\pi}\sin(\pi t)$, $x_2(t) = \frac{1}{\pi}(1 - \cos(\pi))$, $x_3(t) = \frac{t}{\pi} - \frac{\sin(\pi t)}{\pi^2}$. The dual variables become $p_1(t) = -\frac{1}{\pi^2}\cos(\pi t)$, $p_2(t) = -\frac{1}{\pi^2}\sin(\pi t)$ and $p_3(t) = \frac{1}{\pi}$. It should be possible to verify somehow that this solution is actually a global optimum

Without assuming symmetry

To remove the assumption of symmetry we can solve an extended optimization problem. We can still constrain solutions to fulfill $I_{xy} = 0$, since this can always be achieved by rotation of the solution, but we will not assume that the y-axis is a symmetry axis by studying the following optimization problem

$$
\begin{aligned}
&\max \int_0^1 \gamma x_1^2(t) + x_2^2(t)dt - \gamma x_3^2(1) - x_4^2(1) \qquad \text{subject to}\\
&\qquad \dot{x}_1 = u_1, \qquad x_1(0) = 0\\
&\qquad \dot{x}_2 = u_2, \qquad x_2(0) = 0\\
&\qquad \dot{x}_3 = x_1, \qquad x_3(0) = 0\\
&\qquad \dot{x}_4 = x_2, \qquad x_4(0) = 0\\
&\qquad \dot{x}_5 = x_1 x_2, \quad x_5(0) = 0\\
&\quad x_5(1) - x_3(1)x_4(1) = 0 \qquad (I_{xy} = 0)\\
&u_1^2 + u_2^2 \leq 1
\end{aligned}
$$