Continuous-Time Model Identification Using Non-Uniformly Sampled Data

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***Abstract*— This contribution reviews theory, algorithms, and validation results for system identification of continuous-time state-space models from finite input-output sequences. The algorithms developed are autoregressive methods, methods of subspace-based model identification and stochastic realization adapted to the continuous-time context. The resulting model can be decomposed into an input-output model and a stochastic innovations model. Using the Riccati equation, we have designed a procedure to provide a reduced-order stochastic model that is minimal with respect to system order as well as the number of stochastic inputs, thereby avoiding several problems appearing in standard application of stochastic realization to the model validation problem. Next, theory, algorithms and validation results are presented for system identification of continuous-time state-space models from finite non-uniformly sampled input-output sequences. The algorithms developed are methods of model identification and stochastic realization adapted to the continuous-time model context using non-uniformly sampled input-output data.**

I. Introduction

The accurate knowledge of a continuous-time transfer function is a prerequisite to many methods in physical modeling and control system design. System identification, however, is often made by means of time-series analysis applied to discrete-time transfer function models. As yet, there is no undisputed algorithm for parameter translation from discrete-time parameters to a continuous-time description. Problems in this context are associated with translation of the system zeros from the discrete-time model to the continuous-time model whereas the system poles are mapped by means of complex exponentials. As a result, a poor parameter translation tends to affect both the frequency response such as the Bode diagram and the transient response such as the impulse response. Early contributions on continuous-time identification can be found in [21], [3], [22], [23], [10], [18].

In the case of uniform sampling, two circumstances favor the indirect approach via discrete-time identification: Firstly, data are in general available as discrete measurements. Another problem is the mathematical difficulty to treat continuous-time random processes. In the case of non-uniform sampling of data, new problems arise as linear regression based on z-transform properties will fail. The difficulties to convert a discrete-time transfer function to continuous-time transfer function are well known and related to the mapping $f(z) = (\log z)/h$—for non-uniform sampling [6], [5].

In this paper, we derive an algorithm that fits continuous-time transfer function models to discrete-time non-uniformly sampled data and we adopt a hybrid modeling approach by means of a discrete-time disturbance model and a continuous-time transfer function.

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II. A model transformation

This algorithm introduces an algebraic reformulation of transfer function models. In addition, we introduce discrete-time noise models in order to model disturbances. The idea is to find a causal, stable, realizable linear operator that may replace the differential operator while keeping an exact transfer function. This shall be done in such a way that we obtain a linear model for estimation of the original transfer function parameters a_i, b_i. We will consider cases where we obtain a linear model in all-pass or low-pass filter operators. Actually, there is always a linear one-to-one transformation which relates the continuous-time parameters and the convergence points for each choice of operator [12].

Then follows investigations on the state space properties of the introduced filters and the original model. Finally, there are two examples with applications to time-invariant and time-varying systems, respectively. Consider a linear nth order transfer operator formulated with a differential operator $p = d/dt$ and unknown coefficients a_i, b_i.

$$G_0(p) = \frac{b_1 p^{n-1} + \cdots + b_n}{p^n + a_1 p^{n-1} + \cdots + a_n} = \frac{B(p)}{A(p)} \qquad (1)$$

where it is assumed that $A(\cdot)$ and $B(\cdot)$ are coprime. It is supposed that the usual isomorphism between transfer operators and transfer functions, *i.e.*, the corresponding functions of a complex variable s, is valid.

Because of this isomorphism, G_0 will sometimes be regarded as a transfer function and sometimes as a transfer operator. A notational difference will be made with p denoting the differential operator and s denoting the complex frequency variable of the Laplace transform.

On any transfer function describing a physically realizable continuous-time system, it is a necessary requirement that the transfer function be proper because pure derivatives of the input cannot be implemented. This requirement is fulfilled as $\lim_{s\to\infty} G_0(s)$ is finite, *i.e.*, $G_0(s)$ has no poles at infinity. An algebraic approach to system analysis may be suggested. Let a be point on the positive real axis and define the mapping

$$f(s) = \frac{a}{s+a}, \qquad s \in \mathbb{C}$$

Let $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ be the complex plane extended with the 'infinity point'. Then f is a bijective mapping from $\bar{\mathbb{C}}$ to $\bar{\mathbb{C}}$ and it maps the 'infinity point' to the origin and $-a$ to the 'infinity point'. The unstable region—*i.e.*, the right half plane (Re $s > 0$)—is mapped onto a region which does not contain the 'infinity point'. Introduction of the operator

$$\lambda = f(p) = \frac{a}{p+a} = \frac{1}{1+p\tau}, \qquad \tau = 1/a \qquad (2)$$

allows us to make the following transformation

$$G_0(p)_{|p=\frac{1-\lambda}{\tau\lambda}} = G_0^*(\lambda) = \frac{B^*(\lambda)}{A^*(\lambda)}$$

with

$$A^*(\lambda) = 1 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n \qquad (3)$$
$$B^*(\lambda) = \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_n\lambda^n \qquad (4)$$

An input-output model is easily formulated as

$$A^*(\lambda)y(t) = B^*(\lambda)u(t) \qquad (5)$$

or on regression form

$$\begin{aligned} y(t) &= -\alpha_1[\lambda y](t) - \cdots - \alpha_n[\lambda^n y](t) \\ &\quad +\beta_1[\lambda u](t) + \cdots + \beta_n[\lambda^n u](t) \end{aligned} \qquad (6)$$

This is now a linear model of a dynamical system at all points of time. Notice that $[\lambda u], [\lambda y]$ etc. denote filtered inputs and outputs. The parameters α_i, β_i may now be estimated by any suitable method for estimation of parameters of a linear model. A reformulation of the model Eq. (6) to a linear regression form is

$$\begin{aligned} y(t) &= \varphi_\tau^T(t)\theta_\tau, \\ \theta_\tau &= \begin{pmatrix}\alpha_1 & \alpha_2\ldots\alpha_n & \beta_1 & \beta_2\ldots\beta_n\end{pmatrix}^T \\ \varphi_\tau(t) &= (-[\lambda y](t), \ldots -[\lambda^n y](t), \\ &\qquad [\lambda u](t), \ldots [\lambda^n u](t))^T \end{aligned} \qquad (7)$$

with parameter vector θ_τ and the regressor vector φ_τ.

A. Parameter transformations

Before proceeding, we should make clear the relationship between the parameters α_i, β_i of (4) and the original parameters a_i, b_i of the transfer function (1). Let the vector of original parameters be denoted by

$$\theta = \begin{pmatrix}-a_1 & -a_2 & \ldots & -a_n & b_1 & \ldots & b_n\end{pmatrix}^T \qquad (8)$$

Using the definition of λ of Eq. (2), it is straightforward to show that the relationship between the operator-transformed parameters of Eq. (7) and the original parameters of Eq. (8) is

$$\theta_\tau = F_\tau\theta + G_\tau \qquad (9)$$

where the $2n \times 2n-$matrix F_τ is

$$F_\tau = \begin{pmatrix} M_\tau & 0_{n\times n} \\ 0_{n\times n} & M_\tau \end{pmatrix} \qquad (10)$$

and where M_τ is the Pascal matrix

$$M_\tau = \begin{pmatrix} m_{11} & 0 & \cdots & 0 \\ m_{12} & m_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{pmatrix}, \qquad (11)$$

$$m_{ij} = (-1)^{i-j}\binom{n-j}{i-j}\tau^j \qquad (12)$$

Furthermore, the $2n \times 1-$vector G_τ is given by

$$G_\tau = \begin{pmatrix} g_1\ldots g_n & 0\ldots 0\end{pmatrix}^T; \quad g_i = \binom{n}{i}(-1)^i \qquad (13)$$

The matrix F_τ is invertible when M_τ is invertible, *i.e.* for all $\tau > 0$. The parameter transformation is then one-to-one and

$$\theta = F_\tau^{-1}(\theta_\tau - G_\tau) \qquad (14)$$

We may then conclude that the parameters a_i, b_i of the continuous-time transfer function G_0 may be reconstructed from the parameters α_i, β_i of θ_τ by means of basic matrix calculations. As an alternative we may estimate the original parameters a_i, b_i of θ from the linear relation

$$y(t) = \theta_\tau^T\varphi_\tau(t) = (F_\tau\theta + G_\tau)^T\varphi_\tau(t) \qquad (15)$$

where F_τ and G_τ are known matrices for each τ. Moreover, orthogonal linear combinations of the regressor vector components by means of some transformation matrix T could be accommodated by modification of Eq. (15) to

$$y(t) = (T\varphi_\tau(t))^T T^{-T}F_\tau\theta + (T\varphi_\tau(t))^T T^{-T}G_\tau$$

Hence, the parameter vectors θ_τ and θ are related via known and simple linear relationships so that translation between the two parameter vectors can be made without any problem arising. Moreover, identification can be made with respect to either θ or θ_τ.

B. Non-uniform Sampling

Assume that data acquisition has provided finite sequences of non-uniformly sampled input-output data $\{y(t_k)\}_0^N$, $\{u(t_k)\}_0^N$ at sample times $\{t_k\}_0^N$, where $t_{k+1} > t_k$ for all k.

As the regression model of Eq. (6) is valid for all times, it is also a valid regression model at sample times $\{t_k\}_0^N$

$$\begin{aligned} y(t_k) \;=\; & -\alpha_1[\lambda y](t_k) - \cdots - \alpha_n[\lambda^n y](t_k) \\ & +\beta_1[\lambda u](t_k) + \cdots + \beta_n[\lambda^n u](t_k) \end{aligned} \tag{16}$$

Introduce the following brief notation for non-uniformly sampled filtered data

$$[\lambda^j u]_k \;=\; [\lambda^j u](t_k),\ 0 \le j \le n,\ 0 \le k \le N \tag{17}$$

$$[\lambda^j y]_k \;=\; [\lambda^j y](t_k) \tag{18}$$

Introduce the regressor-state vectors

$$x_u = \begin{pmatrix} [\lambda^1 u] \\ [\lambda^2 u] \\ \vdots \\ [\lambda^n u] \end{pmatrix}, \qquad x_y = \begin{pmatrix} [\lambda^1 y] \\ [\lambda^2 y] \\ \vdots \\ [\lambda^n y] \end{pmatrix} \tag{19}$$

with dynamics

$$\tau \dot{x}_u = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \ddots & \ddots & 0 \\ 0 & 1 & -1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} x_u + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \tag{20}$$

$$\tau \dot{x}_y = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \ddots & \ddots & 0 \\ 0 & 1 & -1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} x_u + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} y \tag{21}$$

or

$$\tau \dot{x}_u \;=\; A_\lambda x_u + B_\lambda u, \quad \tau \dot{x}_y = A_\lambda x_y + B_\lambda y \tag{22}$$

Adopting a zero-order-hold (ZOH) approximation, the non-uniformly sampled discretized model will be

$$x_u(t_{k+1}) \;=\; A_k x_u(t_k) + B_k u(t_k) \tag{23}$$

$$x_y(t_{k+1}) \;=\; A_k x_y(t_k) + B_k y(t_k) \tag{24}$$

where

$$A_k \;=\; e^{A_\lambda (t_{k+1}-t_k)/\tau} \tag{25}$$

$$B_k \;=\; \int_0^{(t_{k+1}-t_k)/\tau} e^{A_\lambda s} B_\lambda ds \tag{26}$$

Summarizing the regressor model of Eq. (16) including the regressor filtering, we have

$$\phi(t_k) \;=\; \begin{pmatrix} x_y(t_k) \\ x_u(t_k) \end{pmatrix} \tag{27}$$

$$\phi(t_{k+1}) \;=\; \begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix} \phi(t_k) \tag{28}$$

$$+ \begin{pmatrix} -B_k & 0 \\ 0 & B_k \end{pmatrix} \begin{pmatrix} y(t_k) \\ u(t_k) \end{pmatrix} \tag{29}$$

$$\theta \;=\; \begin{pmatrix} \alpha_1 & \cdots \alpha_n & \beta_1 & \cdots \beta_n \end{pmatrix}^T \tag{30}$$

$$y(t_k) \;=\; \phi(t_k)\theta + w(t_k) \tag{31}$$

where $\{w(t_k)\}$ represents an uncorrelated non-uniformly sampled noise sequence.

III. State-Space Model Identification

Consider a continuous-time time-invariant system $\Sigma_n(A, B, C, D)$ with the state-space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + v(t) \\ y(t) &= Cx(t) + Du(t) + e(t) \end{aligned} \tag{32}$$

with input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, state vector $x \in \mathbb{R}^n$ and zero-mean disturbance stochastic processes $v \in \mathbb{R}^n$, $e \in \mathbb{R}^p$ acting on the state dynamics and the output, respectively. The continuous-time system identification problem is to find estimates of system matrices A, B, C, D from finite sequences $\{u_k\}_{k=0}^N$ and $\{y_k\}_{k=0}^N$ of input-output data. The underlying discretized state sequence $\{x_k\}_{k=0}^N$ and discrete-time stochastic processes $\{v_k\}_{k=0}^N$, $\{e_k\}_{k=0}^N$ correspond to disturbance processes v and e which at the sampling instants can be represented by the discretized components

$$v_k \;=\; \int_{t_{k-1}}^{t_k} e^{A(t_k-s)} v(s) ds,\ k = 1, 2, ..., N \tag{33}$$

$$e_k \;=\; e(t_k) \tag{34}$$

with the covariance $Q \ge 0$, $q = \text{rank}(Q)$

$$\mathcal{E}\{ \begin{bmatrix} v_i \\ e_i \end{bmatrix} \begin{bmatrix} v_j \\ e_j \end{bmatrix}^T \} = Q\delta_{ij} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \delta_{ij}, \tag{35}$$

which enter the non-uniformly sampled and discretized system $\Sigma_n(A_k, B_k, C, D)$ with input $u_k \in \mathbb{R}^m$, output $y_k \in \mathbb{R}^p$, state vector $x_k \in \mathbb{R}^n$ and noise sequences $v_k \in \mathbb{R}^n$, $e_k \in \mathbb{R}^p$ acting on the state dynamics and the output, respectively.

Remark: As computation and statistical validation tests deal with discrete-time data, we assume the original sampled stochastic disturbance sequences to be uncorrelated with a uniform spectrum up to the Nyquist frequency, thereby avoiding the mathematical problems associated with Brownian motion [12].

Continuous-Time State-Space Linear System

From the set of first-order linear differential equations of Eq. (32), one finds the Laplace transforms

$$\begin{aligned} sX &= AX + BU + V + sx_0, \quad x_0 = x(t_0) \\ Y &= CX + DU + E \end{aligned} \tag{36}$$

Introduction of the complex variable transform

$$\lambda(s) = \frac{1}{1+s\tau} \tag{37}$$

corresponding to a stable, causal operator permits an algebraic transformation of the model

$$\begin{aligned} X &= (I+\tau A)[\lambda X] + \tau B[\lambda U] + \tau[\lambda V] + (1-\lambda)x_0 \\ Y &= CX + DU + E \end{aligned} \tag{38}$$

Reformulation while ignoring the initial conditions to linear system equations gives

$$\begin{aligned} \begin{bmatrix} \xi \\ y \end{bmatrix} &= \begin{bmatrix} I+\tau A & \tau B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} \tau v \\ e \end{bmatrix}, \; x(t) = [\lambda \xi](t) \\ &= \begin{bmatrix} A_\lambda & B_\lambda \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} \tau v \\ e \end{bmatrix}, \; \begin{cases} A_\lambda = I + \tau A \\ B_\lambda = \tau B \end{cases} \end{aligned} \tag{39}$$

the mapping between (A, B) and (A_λ, B_λ) being bijective. Provided that a standard positive semi-definiteness condition of Q is fulfilled so that the Riccati equation has a solution, it is possible to replace the linear model of Eq. (39) by the innovations model

$$\begin{bmatrix} \xi \\ y \end{bmatrix} = \begin{bmatrix} A_\lambda & B_\lambda \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} K_\lambda \\ I \end{bmatrix} w, \; K_\lambda = \tau K \tag{40}$$

By recursion it is found that

$$y = Cx + Du + w \tag{41}$$

$$\begin{aligned} &= CA_\lambda[\lambda x] + CB_\lambda[\lambda u] + Du + CK_\lambda[\lambda w] + w \\ &\vdots \\ &= CA_\lambda^k[\lambda^k x] + \sum_{j=1}^{k} CA^{k-j} B_\lambda [\lambda^{k-j} u] + Du \\ &+ \sum_{j=1}^{k} CA^{k-j} K_\lambda [\lambda^{k-j} w] + w \end{aligned} \tag{42}$$

To the purpose of subspace model identification, it is straightforward to formulate extended linear models for the original models and its innovations form

$$\mathcal{Y} = \Gamma_x \mathcal{X} + \Gamma_u \mathcal{U} + \Gamma_v \mathcal{V} + \mathcal{E} \tag{43}$$

$$\mathcal{Y} = \Gamma_x \mathcal{X} + \Gamma_u \mathcal{U} + \Gamma_w \mathcal{W} \tag{44}$$

with state variables $\mathcal{X} = [\lambda^{i-1} x]$ and input-output variables

$$\mathcal{Y} = \begin{bmatrix} [\lambda^{i-1} y] \\ [\lambda^{i-2} y] \\ \vdots \\ [\lambda^1 y] \\ y(t) \end{bmatrix}, \; \mathcal{U} = \begin{bmatrix} [\lambda^{i-1} u] \\ [\lambda^{i-2} u] \\ \vdots \\ [\lambda^1 u] \\ u(t) \end{bmatrix}, \tag{45}$$

and stochastic processes of disturbance

$$\mathcal{V} = \begin{bmatrix} [\lambda^{i-1} v] \\ [\lambda^{i-2} v] \\ \vdots \\ [\lambda^1 v] \\ v(t) \end{bmatrix}, \mathcal{E} = \begin{bmatrix} [\lambda^{i-1} e] \\ [\lambda^{i-2} e] \\ \vdots \\ [\lambda^1 e] \\ e(t) \end{bmatrix}, \mathcal{W} = \begin{bmatrix} [\lambda^{i-1} w] \\ [\lambda^{i-2} w] \\ \vdots \\ [\lambda^1 w] \\ w(t) \end{bmatrix} \tag{46}$$

and parameter matrices of state variables and input-output behavior

$$\Gamma_x = \begin{bmatrix} C \\ CA_\lambda \\ \vdots \\ CA_\lambda^{i-1} \end{bmatrix} \in \mathbb{R}^{ip \times n} \tag{47}$$

$$\Gamma_u = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB_\lambda & D & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ CA_\lambda^{i-2} B_\lambda & CA_\lambda^{i-3} B_\lambda & \cdots & D \end{bmatrix} \in \mathbb{R}^{ip \times im}$$

and for stochastic input-output behavior

$$\Gamma_v = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \tau C & 0 & & 0 & 0 \\ \tau CA_\lambda & \tau C & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ \tau CA_\lambda^{i-2} & \tau CA_\lambda^{i-3} & \cdots & \tau C & 0 \end{bmatrix} \in \mathbb{R}^{ip \times im}$$

and

$$\Gamma_w = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ CK_\lambda & I & \ddots & \vdots & \vdots \\ \vdots & CK_\lambda & \ddots & 0 & 0 \\ CA_\lambda^{i-3} K_\lambda & \vdots & \ddots & I & 0 \\ CA_\lambda^{i-2} K_\lambda & CA_\lambda^{i-3} K_\lambda & \cdots & CK_\lambda & I \end{bmatrix} \tag{48}$$

It is clear that Γ_x of Eq. (47) represents the extended observability matrix as known from linear system theory [20], [19].

System Identification Algorithms

The theory provided permits formulation of a variety of algorithms with the same algebraic properties as the original discrete-time version though with application to continuous-time modeling and identification. Below is presented one realization-based algorithm. Subspace-based algorithms and theoretical justification is to be found in [13].

Algorithm 1 (System realization [8], [13], [9]):

1) Use least-squares identification to find a multivariable transfer function

$$G(\lambda(s)) = D_L^{-1}(\lambda) N_L(\lambda) = \sum_{k=0}^{\infty} G_k \lambda^k \tag{49}$$

where $D_L(\lambda)$, $N_L(\lambda)$are polynomial matrices obtained by means of some identification method such as linear regression with

$$\varepsilon(t, \theta) = D_L(\lambda) y(t) - N_L(\lambda) u(t) \tag{50}$$

$$D_L(\lambda) = I + D_1 \lambda + \cdots + D_n \lambda^n \tag{51}$$

$$N_L(\lambda) = N_0 + N_1 \lambda + \cdots N_n \lambda^n \tag{52}$$

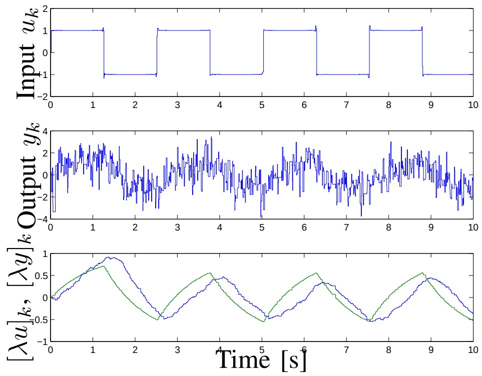

Fig. 1. Non-uniformly sampled data used for continuous-time model identification: Input $\{u_k\}$ (*upper*), output $\{y_k\}$ with stochastic disturbance (*middle*), regressors $\{[\lambda u]_k\}$, $\{[\lambda y]_k\}$ (*lower*).

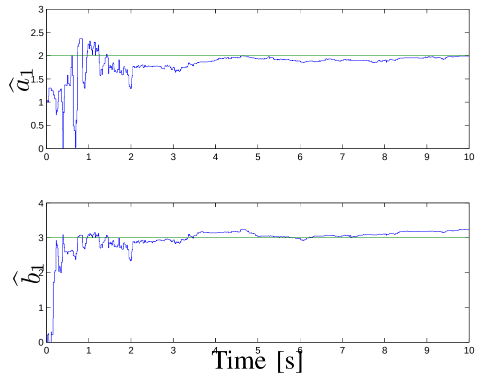

Fig. 2. Continuous-time model identification of Example 1 with $a_1 = 2$, $b_1 = 3$ and recursive least-squares identification using non-uniformly sampled input u and noisy output y. The estimates $\widehat{a}_1$, $\widehat{b}_1$ converge towards the correct values $a_1 = 2$, $b_1 = 3$ ($N = 1000$).

2) Solve for the transformed Markov parameters

$$G_k = N_k - \sum_{j=1}^{k} D_j G_{k-j}, \quad k = 0, \dots, n \quad (53)$$

$$G_k = -\sum_{j=1}^{n} D_j G_{k-j}, \quad k = n+1, \dots, N \quad (54)$$

3) For suitable numbers q, r, s such that $r+s \leq N$ arrange the Markov parameters in the Hankel matrix

$$G_{r,s}^{(q)} = \begin{bmatrix} G_{q+1} & G_{q+2} & \cdots & G_{q+s} \\ G_{q+2} & G_{q+3} & \cdots & G_{q+s+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{q+r} & G_{q+r+1} & \cdots & G_{q+r+s-1} \end{bmatrix} \quad (55)$$

4) Determine rank n and resultant system matrices

$$G_{r,s}^{(0)} = U \Sigma V^T \ \text{(SVD)} \quad (56)$$

$$E_y^T = [I_{p\times p} \quad 0_{p\times(r-1)p}] \quad (57)$$

$$E_u^T = [I_{m\times m} \quad 0_{m\times(s-1)m}] \quad (58)$$

$$\Sigma_n = \text{diag}\ \{\sigma_1, \sigma_2, \dots, \sigma_n\} \quad (59)$$

$$U_n = \text{matrix of first } n \text{ columns of } U \quad (60)$$

$$V_n = \text{matrix of first } n \text{ columns of } V \quad (61)$$

Finally, calculate the state-space matrices

$$A_n = \Sigma_n^{-1/2} U_n^T G_{r,s}^{(1)} V_n \Sigma_n^{-1/2}, \widehat{A} = \frac{1}{\tau}(A_n - I) \quad (62)$$

$$B_n = \Sigma_n^{1/2} V_n^T E_u, \quad \widehat{B} = \frac{1}{\tau} B_n \quad (63)$$

$$C_n = E_y^T U_n \Sigma_n^{1/2}, \quad \widehat{C} = C_n \quad (64)$$

$$D_n = G_0, \quad \widehat{D} = D_n \quad (65)$$

which yields the nth-order state-space realization

$$\begin{aligned} \dot{x}(t) &= \widehat{A}x(t) + \widehat{B}u(t) \\ y(t) &= \widehat{C}x(t) + \widehat{D}u(t) \end{aligned} \quad (66)$$

Algorithm 2 (System realization [8], [13], [9]):

1) Use linear regression to find a truncated multivariable transfer function

$$G^m(\lambda(s)) = \sum_{k=0}^{m} G_k^m \lambda^k \quad (67)$$

where the prediction error

$$\varepsilon(t, \theta) = y(t) - \underbrace{\begin{pmatrix} G_1^m \dots G_2^m \end{pmatrix}}_{\theta} \begin{pmatrix} [\lambda^1 u](t) \\ \vdots \\ [\lambda^m u](t) \end{pmatrix}$$

be minimized at the set of sample times $\{t_k\}_{k=1}^N$ by least-squares estimation of θ or $\{G_k^m\}_{k=1}^m$.

2) For suitable numbers q, r, s such that $r+s \leq N$ arrange the Markov parameters in the Hankel matrix of Eq. (55).

3) Determine rank n and resultant system matrices according to Eqs. (56-61).

4) Finally, calculate the state-space matrices according to Eqs. (62-65) which yields the nth-order state-space realization of Eq. (66).

Remark: A similar algorithm is obtained by replacing Steps 3-4 by balanced model reduction of the system

$$\tau \dot{x}_u = A_\lambda x_u + B_\lambda u, \quad (68)$$

$$y = \widehat{C} x_u, \quad \widehat{C} = \begin{pmatrix} G_1^m \dots G_2^m & 0 \dots 0 \end{pmatrix} \quad (69)$$

with A_λ, B_λ according to Eq. (22).

A. Example: T1DM Blood Glucose Dynamics

Individualized models of blood glucose dynamics are currently of great interest for improved clinical therapy for Type-1 diabetes (T1DM) patients [17], [1], [2], [16]. To this purpose, a clinical protocol for data acquisition was designed under the aegis of DIAdvisor [1], a large scale FP7-IST European project, reviewed and approved by the ethical committees of the Clinical Investigation Centers participating in the trials, namely, Montpellier University Hospital (CHU) in Montpellier, France, Padova University Clinics (UNIPD) in Padova, Italy and the Clinical Institute of Experimental Medicine (IKEM) in Prague, Czech Republic. A population of T1DM subjects using basal-bolus insulin regimen participated in the study, signing an informed and witnessed consent form. The trial comprised a series of intermittent experiment sessions for a duration of up to 9 weeks per patient. Patients were admitted to the clinic for a 6.5 hours observation period, from 6:30 am to 1:00 pm, fasting from the midnight, equipped with a Dexcom Seven®Plus continuous glucose monitoring sensor (CGMS) for interstitial glucose samples and a HemoCue Glucose 201+ Analyzer for capillary blood glucose measurements [4], [7]. After arrival, a recalibration of the CGM system was performed by the subjects using the HemoCue meter, in order to be able to start data collection at 7:00 with a well calibrated glucose monitoring device. A standardized breakfast, the amount of carbohydrate being 40 [g], was served at 8:00am and fully ingested within 20 minutes. Insulin was injected at 10:00am.

Application of Algorithm 2 was successful in accurate modeling of the blood glucose concentration response to meals and insulin. Figure 3 shows an example of non-uniformly sampled diabetic blood glucose concentration $y(t)$, continuous-time model output $\widehat{y}(t)$, model error $y(t)-\widehat{y}(t)$, in response to food input u_g, insulin input u_i with regressors for $\tau = 10$ [min], $n = 4$ and $m = 10$. The error $\mathcal{L}_2$ norm of the open-loop model response to inputs was less than 1% of the output $\mathcal{L}_2$ norm for model order $n = 4$.

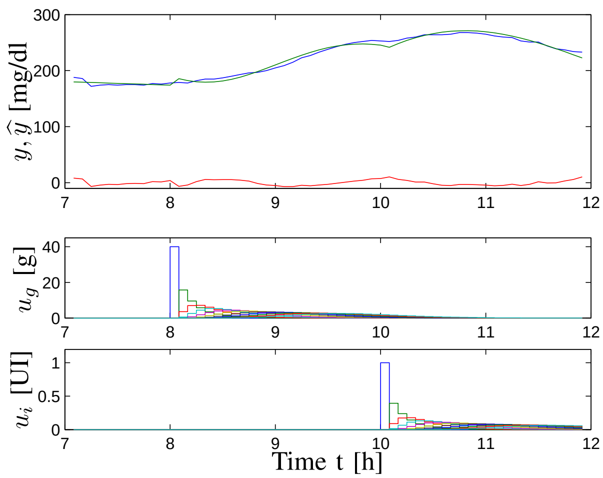

Fig. 3. Non-uniformly sampled diabetic blood glucose concentration $y(t)$ (*upper, blue*), continuous-time model output $\widehat{y}(t)$ (*upper, green*), model error $y(t) - \widehat{y}(t)$ (*upper, red*), in response to food input u_g (*middle*), insulin input u_i (*lower*) with regressors for $\tau = 10$ [min], $n = 4$ and $m = 10$.

IV. Discussion and Conclusions

We have formulated an identification method for continuous-time state-space models using non-uniformly sampled data [11]. The transformation by means of λ allows an *exact* reparametrization of a continuous-time transfer function. High-frequency dynamics and low-frequency dynamics thus appear without distortion in the mapping from input to output. Both the operator translation and filtering approaches such as the Poisson moment functional (PMF) or the Laguerre polynomials give rise to similar estimation models for the deterministic case [18], [22], [23]. Implementation of the operator λ may be done as continuous-time filters, discrete-time filters or by means of numerical integration methods [12]. Whereas ZOH only was studied here, inter-sample behavior is significant for approximation properties.

The main differences between this method and previous approaches to continuous-time model identification consist of a different estimation model and a new parametrization of the continuous-time transfer function whereas the parameter estimation method are standard methods [12]. Analysis of convergence and statistical consistency was presented in [12].

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