On Resilience of Multicommodity Flow Networks

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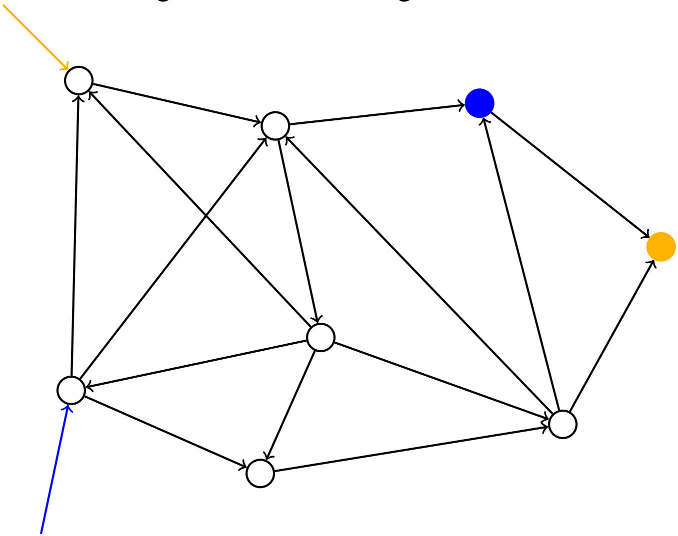
Multicommodity Flow Problem

Heterogeneous flows through common links



In this talk: Introduce dynamics to study stability and resilience.

Multicommodity Flows

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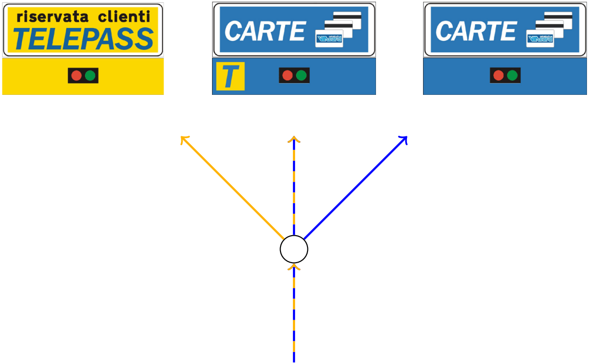
Outline

Example - Toll Station

Images: `http://www.inabruzzo.com`, `http://www.autostrade.it`

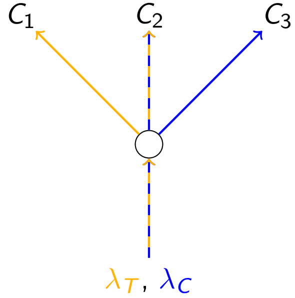
Example - Toll Station

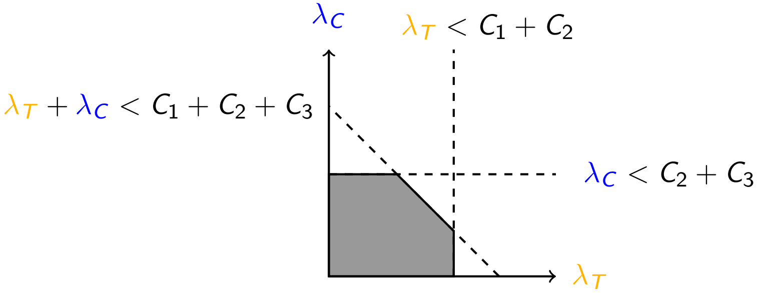


λ_T, λ_C

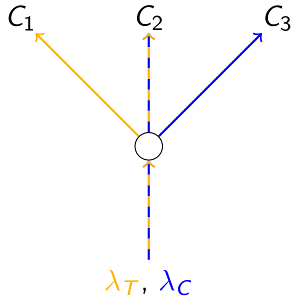
Static inflow of customers with telepass, λ_T, and credit card, λ_C.

Example - Stability Region





Local Network - Dynamics



ρ_i^k density of commodity k on link i

$\rho_i = \sum_k \rho_i^k$ aggregate density

$G_i^k(\rho_1, \rho_2, \rho_3)$ splitting policy

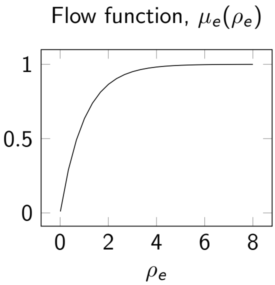
$\mu_i(\rho_i)$ flow function, $C_i = \sup_{\rho_i} \mu_i(\rho_i)$

$$\dot{\rho}_i^k = \lambda_k G_i^k(\rho_1(t), \rho_2(t), \rho_3(t)) - \frac{\rho_i^k(t)}{\rho_i(t)} \mu_i(\rho_i(t))$$

Local Network - Assumptions

Flow functions

Strictly increasing and bounded from above.



Splitting policies

a) $\dfrac{\partial}{\partial \rho_j} G_i^k(\rho) \geq 0, \quad \forall i, j \text{ s.t. } i \neq j.$

b) if $\rho_i \to +\infty$ on a subset of links, then $G_i^k(\rho) \to 0$.

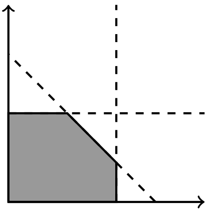
Local Network - Stability

Theorem

For a local network, satisfying the assumptions on the previous slide, it holds that:

a) *if the inflows are inside the stability region, it exists unique limit densities, ρ_e^{k*}, such that $\lim_{t \to +\infty} \rho_e^k(t) = \rho_e^{k*}$ for every link.*

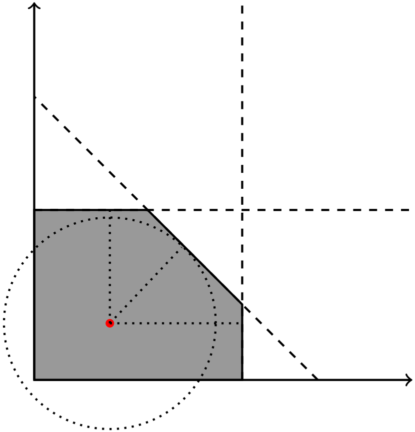
b) *if the inflows are outside the stability region, $\rho_e(t) \to \infty$ on at least one link.*



Proof idea: The system is monotone in the aggregate

Resilience

The largest increase of inflow/decrease of capacity the network can handle

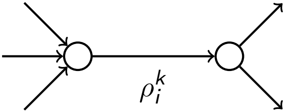


Margin of resilience =

$$\min\{C_1 + C_2 + C_3 - \lambda_C - \lambda_T,\ C_1 + C_2 - \lambda_T,\ C_2 + C_3 - \lambda_C\}$$

Outline

Global Network - Dynamics



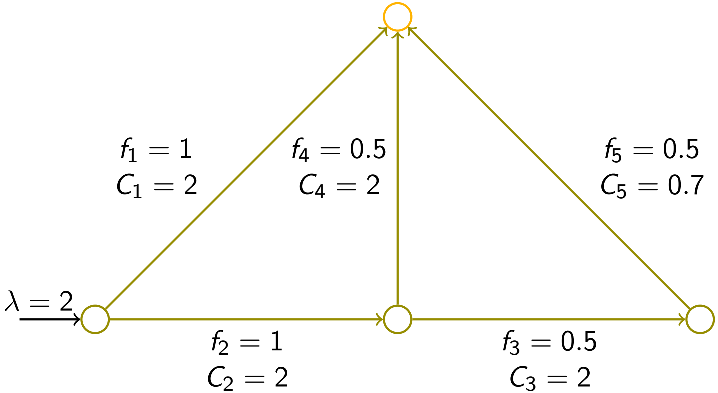
$$\dot{\rho}_i^k = \sum_j f_{ji}^k - \sum_j f_{ij}^k$$

$$f_{ij}^k = \begin{cases} \frac{\rho_i^k}{\rho_i} \mu_i(\rho_i) G_j^k(\rho) & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{o.w} \end{cases}$$

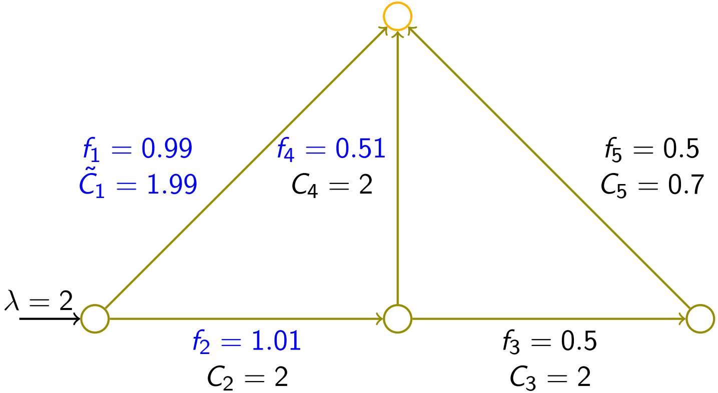
Proposition

For acyclic networks, where the splitting policies and flow functions are satisfying the previously stated assumptions, if there exists a finite equilibrium, it is globally asymptotically stable.

Resilience - Single Commodity

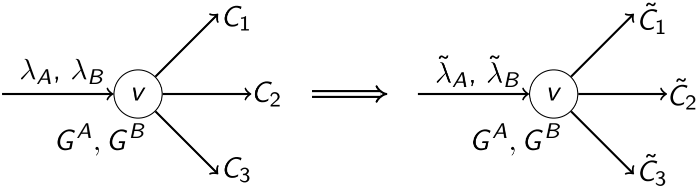


Resilience - Single Commodity



Margin of resilience = Minimum node residual capacity = 0.2
[Como et.al. (2013), *Robust Distributed Routing in Dynamical Networks - Part I & II*]

Resilience - Diffusivity

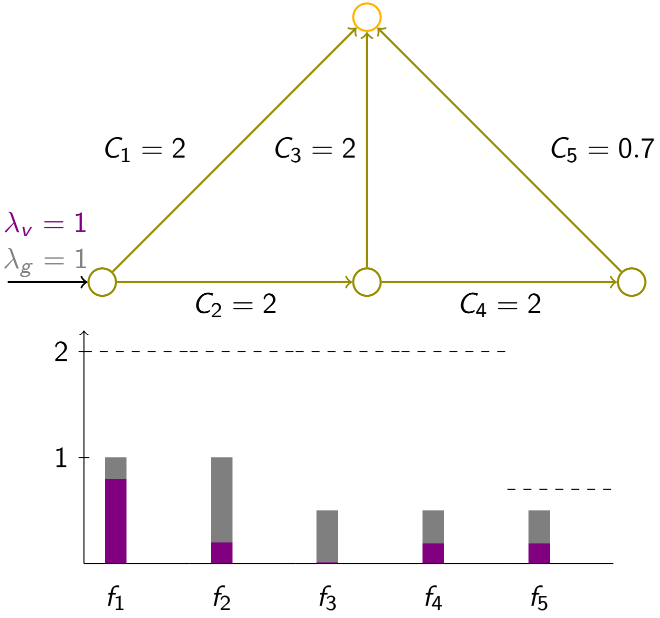


Proposition

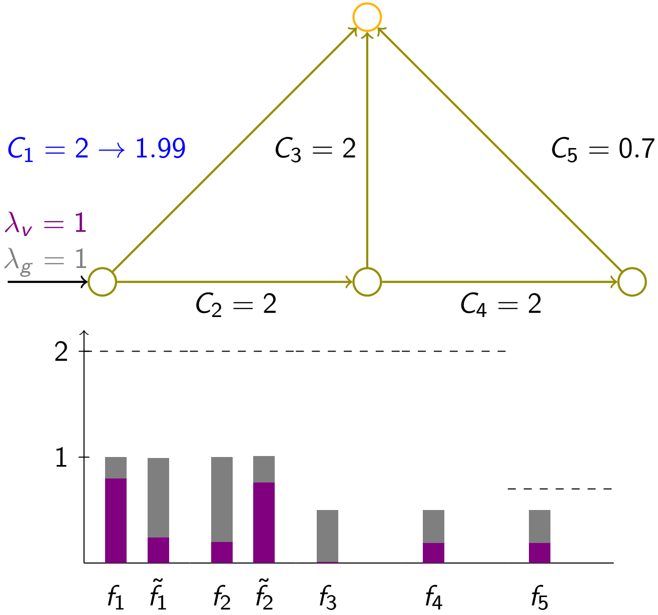
For every subset of links $\mathcal{I}$ it holds that

$$\sum_{i\in\mathcal{I}}\left(\tilde{f}_i^*(\tilde{\lambda})-f_i^*\right)\leq\sum_{k\in\mathcal{K}}\left[\tilde{\lambda}_k-\lambda_k\right]_+ +\sum_e(C_e-\tilde{C}_e).$$

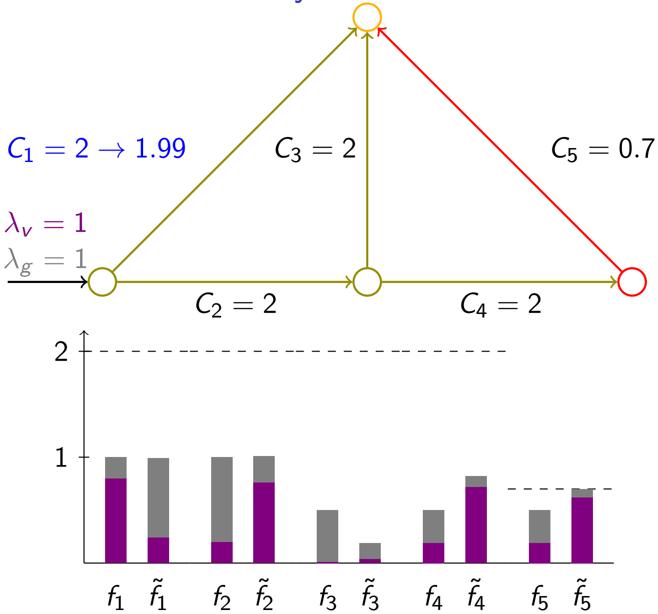
Resilience - Multicommodity



Resilience - Multicommodity



Resilience - Multicommodity



Margin of resilience ≈ 0

Outline

Conclusions and Future Work

Conclusions

- Extension of single commodity to multicommodity
- Heterogeneity in the routing can make the network more fragile

Future Work

- Resilience under less heterogeneous routing policies
- Robust (distributed) controllers, scheduling

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