

## ANYTIME RELIABLE TRANSMISSION OF REAL-VALUED INFORMATION THROUGH DIGITAL NOISY CHANNELS\*

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**Abstract.** The problem of reliably transmitting a real-valued random vector through a digital noisy channel is relevant for the design of distributed estimation and control techniques over networked systems. One important example consists in the remote state estimation under communication constraints. In this case, an anytime transmission scheme consists of an encoder—which maps the real vector into a sequence of channel inputs—and a decoder—which sequentially updates its estimate of the vector as more and more channel outputs are observed. The encoder performs both source and channel coding of the data. Assuming that no channel feedback is available at the transmitter, this paper studies the rates of convergence to zero of the mean squared error. Two coding strategies are analyzed: the first one has exponential convergence rate but is expensive in terms of its encoder/decoder computational complexity, while the second one has a convenient computational complexity but subexponential convergence rate. General bounds are obtained describing the convergence properties of these classes of methods.

**Key words.** state estimation with communication constraints, anytime transmission, real-time communication, unequal error protection, digital fountain codes

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**1. Introduction.** Reliable transmission of information among the nodes of a network is known to be a relevant problem in information engineering. It is indeed fundamental when the network is designed for pure information transmission, as well as in scenarios in which the network is deputed to accomplish some specific tasks requiring information exchange. Important examples include networks of processors performing parallel and distributed computation [2, 42], or load balancing [13, 14, 30]; wireless sensor networks, in which the final goal is estimation and decision making from distributed measurements [19, 21, 45, 15]; sensors/actuators networks, such as mobile multiagent networks, in which the final goal is control [20, 32, 29, 33]. Distributed algorithms to accomplish synchronization, estimation, or localization tasks necessarily need to exchange quantities among the agents, which are often real-valued. Assuming that transmission links are digital, a fundamental problem is thus to transmit a continuous quantity, i.e., a real number or, possibly, a vector, through a digital noisy channel up to a certain degree of precision.

This paper is concerned with the problem of efficient, real-time transmission of a finite-dimensional Euclidean-space-valued state through a noisy digital channel. We shall focus on anytime transmission algorithms, i.e., algorithms which can be stopped

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anytime while providing estimations of increasing precision. These algorithms are particularly suitable for applications in problems of distributed control.

As especially pointed out in a series of works by Sahai and Mitter [36, 35, 37], there is a specific feature distinguishing the problem of information transmission for control from the problem of pure information transmission. This is related to the different sensitivity to delay typically occurring in the two scenarios. Indeed, while the presence of sensible delays can often be tolerated in the communication performance evaluation, such delays can be detrimental for the system performance in several control applications. Here, the fundamental question is not only where, but also when, the information is available. For this reason, it is often desirable to use transmission systems for control applications which are able to provide estimates whose precision increases with time, so as providing a reasonable partial information transmission anytime the process is stopped.

On the other hand, the computational complexity of the transmission schemes is a central issue. In fact, nodes in wireless networks are usually very simple devices with limited computational abilities and severe energy constraints. Applicable transmission systems should be designed performing a number of operations which remains bounded in time, both in the encoding and in the decoding. Hence, an analysis of the trade-offs between performance and complexity of the transmission schemes is required.

In many problems of information transmission, it is possible to take advantage of the feedback information naturally available to the transmitter. Known results in information theory [9] show that feedback can improve the capacity of channels with memory, or multiple access channels,<sup>1</sup> as well as reduce latency and computational complexity. In many cases of practical interest, however, feedback information is incomplete or difficult to use. Also, there are many situations, for instance, in the wireless network scenario, in which the transmitter needs to broadcast its information to many different receivers, and hence feedback strategies to acknowledge the receipt of past transmissions could be infeasible. For these reasons, in the present paper we shall restrict ourselves to the case in which there is no feedback information.

A fundamental characteristic of digital communication for control applications concerns the nature of information bits. In the traditional communication theory, information bits are usually assumed to be equally valuable, and they are consequently given the same priority by the transmission system designer. In fact, design paradigms of modern low-complexity codes [28, 34]—based on random sparse graphical models and iterative decoding algorithms—treat information bits as equally valuable. While such an assumption is typically justified by the source-channel separation principle, this principle does not generally hold when delay is a primary concern. For instance, it is known that separate source-channel coding is suboptimal in terms of the joint source-channel error exponent [10, 11]. In fact, in many problems of information transmission for control or estimation, different information bits typically require significantly different treatment.

As an example, particularly relevant for the topics addressed in this paper, assume that a random parameter, uniformly distributed over a unitary interval, has to be reliably transmitted through a digital noisy channel (see [5] and the references therein for the analysis of the information theoretic limits of this problem on the bandwidth-unlimited Gaussian channel). Such a parameter may be represented by its dyadic expansion, which is a stream of independent and identically distributed bits.

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<sup>1</sup>However, a classic result due to Shannon shows that feedback does not improve the capacity of a discrete memoryless channel.

Clearly, such information bits are not equally valuable, since the first one is more significant than the second one, and so on. This motivates the study of unequal error protection codes [23, 4]. One of the challenges posed by information transmission for control/estimation applications is to come up with design paradigms for practical, low-complexity, unequal error protection codes.

In this paper, we shall propose two classes of coding strategies for the anytime transmission of real-valued random vectors through a digital noisy channel. In both cases, the transmission scheme consists of an encoder, mapping the real vector into a sequence of channel inputs, and of a decoder, sequentially refining the estimate of the vector as more and more channel outputs are observed. The first strategy is characterized by good performance in terms of the convergence of the mean squared error, but it is expensive in terms of encoder/decoder computational complexity. On the other hand, the second class of strategies has convenient computational complexity but worse convergence rates.

In order to keep the use of information-theoretical techniques at a minimum, we shall confine our exposition to the binary erasure channel (BEC) and defer any discussion on the possible extensions to general discrete memoryless channels to the concluding section. In the BEC, a transmitted binary signal is either correctly received or erased with some probability  $\varepsilon$ . While this channel allows for an elementary treatment, it is of its own interest in many scenarios. In [6], the techniques proposed here have been applied in order to obtain a version of the average consensus algorithm working in the presence of digital erasure communication channels between the nodes.

The rest of this paper is organized as follows. Section 2 formally states the problem. Section 3 presents an upper bound to the possible error convergence of any coding scheme over the BEC. In section 4 we introduce the class of encoder/decoder schemes used throughout the paper which are based on a preliminary vector quantization of the continuous vector to be transmitted. In section 5, trade-offs between performance and computational complexity are investigated. First, a simple linear-time encodable/decodable repetition scheme is analyzed in section 5.1. Then the main result is presented in section 5.2, showing that finite-window coding schemes are able to achieve only subexponential error decays. In section 6, random linear convolutional codes are shown to achieve exponential error rates at the cost of computational complexity growing quadratically in time. Finally, some Monte Carlo simulations of finite-window coding schemes are reported in section 7.

We end this introduction by establishing some notation. Throughout the paper,  $\mathbb{R}$  and  $\mathbb{N}$  will denote the sets of reals and naturals, respectively. For a subset  $A \subseteq B$ ,  $|A|$  will denote the cardinality of  $A$ ,  $\overline{A} = B \setminus A$  its complement, and  $\mathbb{1}_A : B \rightarrow \{0, 1\}$  its indicator function, defined by  $\mathbb{1}_A(x) = 1$  if  $x \in A$ , and  $\mathbb{1}_A(x) = 0$  otherwise. The natural logarithm will be denoted by  $\ln$ , while  $\log$  will stand for the logarithm in base 2. For  $x \in [0, 1]$ , we shall use the notation  $H(x) := -x \log x - (1 - x) \log(1 - x)$  for the binary entropy of  $x$  with the standard convention  $0 \log 0 = 0$ . For two sequences of reals  $(a_t)_{t \in \mathbb{N}}$  and  $(b_t)_{t \in \mathbb{N}}$ , both the notations  $a_t = O(b_t)$  and  $b_t = \Theta(a_t)$  will mean that  $a_t \leq K b_t$  for some constant  $K$ , while  $a_t = o(b_t)$  will mean that  $\lim_t a_t/b_t = 0$ . A sequence  $a_t, t = 1, 2, \dots$ , is sometimes denoted with the symbol  $a = (a_t)_{t=1}^\infty$ , while the symbol  $a = (a_t)_{t=1}^T$  will mean its truncation to  $t = 1, \dots, T$ .

**2. Problem formulation.** We shall now provide a formal description of the problem. Let  $x$  be a random variable taking values on  $\mathcal{X} \subseteq \mathbb{R}^d$ . We shall assume that  $x$  has an a priori probability law which is absolutely continuous with respect to the Lebesgue measure, and denote by  $f(x)$  the probability density of  $x$ . Further, we

shall assume that  $\mathbb{E}\|x\|^{2+\delta} < +\infty$  for some  $\delta > 0$ . At time  $t \in \mathbb{N}$ , the communication channel has input  $y_t$  and output  $z_t$ , taking values in some finite alphabets  $\mathcal{Y}$  and  $\mathcal{Z}$ , respectively. Transmission is assumed to be memoryless; i.e., given the current input  $y_t$ , the output  $z_t$  is assumed to be conditionally independent from the previous inputs  $(y_s)_{s=1}^{t-1}$  and outputs  $(z_s)_{s=1}^{t-1}$ , as well as from the vector  $x$ . The conditional probability of  $z_t = z$  given  $y_t = y$  will be assumed stationary and denoted by  $p(z|y)$ . We shall consider in detail the binary erasure channel (BEC) in which  $\mathcal{Y} = \{0, 1\}$ ,  $\mathcal{Z} = \{0, 1, ?\}$ , and

$$p(?|0) = p(?|1) = \varepsilon, \quad p(0|0) = p(1|1) = 1 - \varepsilon, \quad p(1|0) = p(0|1) = 0.$$

Here, ? stands for the erased signal and  $\varepsilon \in [0, 1[$  for the erasure probability.

The anytime transmission scheme consists of an encoder and a sequential decoder.<sup>2</sup> The encoder consists of a family of maps  $E_t : \mathcal{X} \rightarrow \mathcal{Y}$  specifying the symbol transmitted through the channel at time  $t$ ,  $y_t = E_t(x)$ . With this family of maps we can associate the global map  $\mathcal{E} : \mathcal{X} \rightarrow \mathcal{Y}^{\mathbb{N}}$  which specifies the infinite string that the encoder generates from  $x$ . The decoder instead is given by a family of maps  $\mathcal{D}_t : \mathcal{Z}^t \rightarrow \mathcal{X}$ , describing the estimate  $\hat{x}_t = \mathcal{D}_t((z_s)_{s=1}^t)$  of  $x$  obtained from the string  $(z_s)_{s=1}^t$  that has been received until time  $t$ . With this family of maps we can associate naturally the global map  $\mathcal{D} : \mathcal{Z}^{\mathbb{N}} \rightarrow \mathcal{X}^{\mathbb{N}}$ . This is represented in the scheme

$$(2.1) \quad \begin{array}{ccccccc} \mathcal{X} & \xrightarrow{\mathcal{E}_t} & \mathcal{Y}^t & \xrightarrow{\text{Channel}} & \mathcal{Z}^t & \xrightarrow{\mathcal{D}_t} & \mathcal{X} \\ x & \longmapsto & (y_s)_{s=1}^t & \longmapsto & (z_s)_{s=1}^t & \longmapsto & \hat{x}_t, \end{array}$$

where  $\mathcal{E}_t := \pi_t \circ \mathcal{E}$  and where  $\pi_t : \mathcal{Y}^{\mathbb{N}} \rightarrow \mathcal{Y}^t$  is the projection of a sequence in  $\mathcal{Y}^{\mathbb{N}}$  into its first  $t$  symbols

In order to evaluate the performance of a scheme, we define the root mean squared error (mean with respect to both the randomness of  $x \in \mathcal{X}$  and with respect to the possible randomness of the communication channel) at time  $t$  by

$$(2.2) \quad \Delta_t := (\mathbb{E}\|x - \hat{x}_t\|^2)^{1/2}.$$

In this paper, we shall be concerned with the rate of decay of  $\Delta_t$  for different anytime transmission schemes. All the coding strategies which will be analyzed are characterized by a root mean squared error  $\Delta_t$  converging to zero like  $2^{-\beta t^\alpha}$  for some constants  $\beta > 0$  and  $0 < \alpha \leq 1$ . More precisely we shall seek to find  $\alpha, \beta$  such that

$$(2.3) \quad \Delta_t \leq p(t)2^{-\beta t^\alpha}$$

for some polynomial  $p(t)$ . When (2.3) holds, the coding strategy will be said to achieve a degree of convergence  $\alpha$  and rate of convergence  $\beta$ . When  $\alpha = 1$  we shall simply say that we have an exponential convergence. In this case  $\beta$  is referred to as the exponential convergence rate. In what follows, various strategies will be compared in terms of the parameters  $\alpha$  and  $\beta$  that can be achieved, and such parameters will be related to the required computational complexity.

<sup>2</sup>Our definition of anytime transmission scheme does not formally coincide with that in the anytime information theory of Mitter and Sahai. Our usage of the term “anytime” has to be understood in the broader artificial intelligence sense, where anytime algorithms are algorithms whose quality of results improves gradually as computation time increases [46].

**2.1. Application to state estimation under communication constraints.**

The problem illustrated in the previous paragraph is related to the state estimation problem under communication constraints (see [39, 40, 25, 38, 26, 24, 27] and the references therein). Assume we are given a discrete time stochastic linear system

$$(2.4) \quad x(t + 1) = Ax(t) + v(t), \quad x(0) = x_0,$$

where  $x_0 \in \mathbb{R}^n$  is a random vector with zero mean,  $v(t) \in \mathbb{R}^n$  is a zero-mean white noise,  $x(t) \in \mathbb{R}^n$  is the state sequence, and  $A \in \mathbb{R}^{n \times n}$  is a full rank, unstable matrix.

Suppose that a remotely positioned receiver is required to estimate the state of the system, while observing the output of a binary erasure channel only. Then it is necessary to design a family of encoders  $E_t$  and of decoders  $D_t$ . At each time  $t \geq 0$ , the encoder  $E_t$  takes  $x(0), \dots, x(t)$  as input and returns the symbol  $y_t \in \{0, 1\}$ , which is in turn fed as an input to the channel. The receiver observes the channel output symbols  $z_0, \dots, z_t$ , from which the decoder  $D_t$  has to obtain an estimate  $\hat{x}(t)$  of the current state.

If we have that  $v(t) = 0$  for every  $t \geq 0$ , then the only source of uncertainty is due to the initial condition  $x_0$ . Hence, in this case, the encoder/decoder task reduces to obtaining good estimates of  $x_0$  at the receiver side. Indeed, in order to obtain a good estimate  $\hat{x}(t)$  of  $x(t)$ , the receiver has to obtain the best possible estimate  $\hat{x}(0|t)$  of the initial condition  $x(0)$  from the received data  $y_0, \dots, y_t$ , and then it can define  $\hat{x}(t) := A^t \hat{x}(0|t)$ . In this way, one has  $x(t) - \hat{x}(t) = A^t(x(0) - \hat{x}(0|t))$ , so that the problem reduces to finding the best way of coding  $x(0)$  in such a way that expansion of  $A^t$  is well dominated by the contraction of  $x(0) - \hat{x}(0|t)$ . The same technique can be applied if  $v(t)$  is small with respect to  $x_0$ , as clarified by the following example.

*Example 1.* Consider the unstable scalar discrete time linear system

$$x(t + 1) = ax(t) + v(t), \quad x(0) = x_0,$$

where  $a > 1$ ,  $x_0$  is a random variable with probability density  $f(x)$ , and  $v(t)$  is a sequence of independent and identically distributed random variables with zero mean and variance  $\sigma_v^2$ , which are independent of  $x_0$ . Assume that a state estimation algorithm is run, based on the noiseless model  $x(t + 1) = ax(t)$ , by estimating the initial condition  $x_0$  from data transmitted until time  $t$ . As before, we shall denote this estimate by  $\hat{x}(0|t)$ . From  $\hat{x}(0|t)$ , we form the estimate  $\hat{x}(t) := a^t \hat{x}(0|t)$  of  $x(t)$ . The estimation error at time  $t$  will be  $e(t) := x(t) - \hat{x}(t) = a^t(x(0) - \hat{x}(0|t)) + \sum_{i=0}^{t-1} a^{t-1-i} v(i)$ , so that  $\mathbb{E}[e(t)^2] = a^{2t} \mathbb{E}[(x(0) - \hat{x}(0|t))^2] + \sigma_v^2 \frac{1-a^{2t}}{1-a^2}$ . This error depends both on the error in the estimation of the initial condition and on the wrong model we used. As we shall see, our techniques yield an estimation error on  $x(0)$  of the form  $\mathbb{E}[(x(0) - \hat{x}(0|t))^2] = C\zeta(t)$ , where  $C$  depends only on the probability density  $f(x)$  and  $\zeta(t)$  is a function converging to zero depending only on the communication channel characteristics and on the coding strategy. Therefore,

$$\mathbb{E}[e(t)^2] = a^{2t} \left[ C\zeta(t) + \sigma_v^2 \frac{1 - a^{-2t}}{a^2 - 1} \right].$$

In case  $C$  is much larger than  $\sigma_v^2$ , there will be an initial time regime in which the error is not influenced by the model noise but only by the estimation of the initial condition  $x(0)$ .

**3. The limit of performance on the BEC.** Observe that, in case of a noiseless channel, the function mapping  $x$  into  $(\hat{x}_0, \dots, \hat{x}_{t-1})$  is a quantizer assuming at most

$2^t$  values. It is well known in the theory of vector quantization [18] that if  $\mathcal{Q} : \mathcal{X} \rightarrow \mathcal{X}$  is a quantizer assuming  $m$  values, then

$$(3.1) \quad (\mathbb{E} \|x - \mathcal{Q}(x)\|^2)^{1/2} \geq C_- m^{-1/d},$$

where  $C_-$  is a positive constant depending only on the dimension  $d$  and the a priori density  $f(x)$ . This shows that  $\Delta_t \geq C_- 2^{-t/d}$  for all  $t \in \mathbb{N}$ . Hence, it is not possible to obtain a convergence degree  $\alpha$  greater than 1 with an exponential convergence rate  $\beta$  larger than  $1/d$ . In this section, we shall present a tighter upper bound on the exponential convergence rate of  $\Delta_t$  on the BEC with erasure probability  $\varepsilon$ .

Consider the general scheme (2.1). The error pattern associated to the output sequence  $(z_t) \in \mathcal{Z}^{\mathbb{N}}$  is the sequence  $(\xi_t) \in \{c, ?\}^{\mathbb{N}}$  componentwise defined by  $\xi_t = c$  if  $z_t \in \{0, 1\}$  (this corresponds to a correct transmission), and  $\xi_t = ?$  if  $z_t = ?$  (this corresponds to an erased signal). Observe that, given the encoder  $\mathcal{E}$  and the decoder  $\mathcal{D}$ , the error pattern  $(\xi_t)_{t \in \mathbb{N}}$  is a random variable independent of the source vector  $x$ . This property will allow us to present for the BEC almost elementary proofs of results holding true also for general discrete memoryless channels.

For  $j \leq t$ , let

$$(3.2) \quad \lambda_j^t := \sum_{j \leq s \leq t} \mathbb{1}_{\{\xi_s = c\}}$$

be the random variable describing the number of nonerased outputs observed between time  $j$  and  $t$ . Clearly,

$$(3.3) \quad \mathbb{P}(\lambda_j^t = l) = \binom{t-j+1}{l} \varepsilon^{t-j+1-l} (1-\varepsilon)^l, \quad l = 0, \dots, t-j+1.$$

The simple observation above allows one to prove the following result.

**THEOREM 3.1.** *Assume transmission over the BEC with erasure probability  $\varepsilon \in [0, 1]$ . Then the estimation error of any coding scheme as in (2.1) satisfies*

$$(3.4) \quad \Delta_t \geq C_- 2^{-t\bar{\beta}(d,\varepsilon)}$$

for all  $t \geq 0$ , where

$$(3.5) \quad \bar{\beta}(d, \varepsilon) := -\frac{1}{2} \log \left( \varepsilon + (1-\varepsilon)2^{-2/d} \right)$$

and  $C_-$  is a constant depending only on the dimension  $d$  and the a priori density  $f(x)$ .

*Proof.* Conditioned on the infinite error pattern  $(\xi_s)_{s \in \mathbb{N}}$ , the channel reduces to a deterministic map, so that the composition of all the maps in (2.1) becomes a quantizer from  $\mathcal{X}$  to itself with a range of cardinality not larger than  $2^{\lambda_1^t}$ . From this fact and from (3.1), we can deduce that  $\mathbb{E} [\|x - \hat{x}_t\|^2 | \lambda_1^t = l] \geq C_-^2 2^{-2l/d}$ . Therefore,

$$(3.6) \quad \begin{aligned} \mathbb{E} [\|x - \hat{x}_t\|^2] &= \sum_{l=0}^t \mathbb{E} [\|x - \hat{x}_t\|^2 | \lambda_1^t = l] \mathbb{P}(\lambda_1^t = l) \\ &\geq C_-^2 \sum_{l=0}^t 2^{-2l/d} \binom{t}{l} \varepsilon^{t-l} (1-\varepsilon)^l \\ &= C_-^2 (\varepsilon + (1-\varepsilon)2^{-2/d})^t. \end{aligned}$$

From the inequality above, the claim follows.  $\square$

*Remark 1.* The Shannon capacity of the BEC (measured in bits per channel use) equals  $1 - \varepsilon$ , which is the average number of nonerased bits per channel use. It can be directly verified that<sup>3</sup>

$$(3.7) \quad \bar{\beta}(d, \varepsilon) < \frac{1}{d}(1 - \varepsilon) \quad \forall \varepsilon \in ]0, 1[.$$

Inequality (3.7) shows that the estimation error of any coding scheme after  $t$  uses of a digital noisy channel is exponentially larger than that of a quantizer whose image has cardinality  $(1 - \varepsilon)t$ . In fact, the latter is the lower bound one would have obtained by simply using a rate distortion argument. Indeed, a closer look at (3.6) reveals that the second summation is asymptotically dominated by the term corresponding to  $l = l_t^* := \lfloor t \frac{1-\varepsilon}{2^{1/d} \varepsilon + 1 - \varepsilon} \rfloor$ , while the average number of unerased bits is given by  $\mathbb{E}[\lambda_1^t] = (1 - \varepsilon)t$ . Hence, the exponential rate is dominated by atypical channel realizations, i.e., by the events  $\{\lambda_1^t = l_t^*\}$  of probability exponentially vanishing in  $t$ .

It is not hard to see that (3.4) continues to hold true even if the encoder has access to noiseless (even noncausal) output feedback.<sup>4</sup> A fortiori, (3.4) holds in the case of partial or noisy feedback, which is the typical situation occurring in the network scenarios outlined in section 1. In the case of perfect causal feedback, the bound (3.4) is achieved by the encoder, which keeps on transmitting the most significant bit of the dyadic expansion (see section 4) of  $x$  until this is correctly received. However, it is not clear what can be done if the feedback is noisy, partial, or not available (as in the applications outlined in section 1). In section 5.1 we shall propose some simple schemes which are not able to achieve exponential error rates, but have low computational complexity, while in section 6 we shall present schemes achieving exponential error rates at the cost of higher computational complexity.

**4. Quantized encoding schemes.** In this paper, we shall propose and compare different coding strategies. All of them are based on a separation between the quantization of the continuous vector and the channel coding. In the literature, vector quantizers with special structure have been proposed, called tree-structured vector quantizers [18]. Consider a map  $\mathcal{S} : \mathcal{X} \rightarrow \{0, 1\}^{\mathbb{N}}$ , and, for all  $t \in \mathbb{N}$ , the map  $\mathcal{S}_t := \pi_t \circ \mathcal{S}$ , where  $\pi_t : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^t$  is the truncation operator defined above. Finally, let  $\mathcal{S}_t^{-1}$  be a right inverse of  $\mathcal{S}_t$ . Then we can define a tree-structured vector quantizer [18, p. 410] which is the family of maps  $\mathcal{Q}_t : \mathcal{X} \rightarrow \mathcal{X}$  defined as  $\mathcal{Q}_t := \mathcal{S}_t^{-1} \circ \mathcal{S}_t$ . It can be seen [18] that if  $\mathbb{E}\|x\|^{2+\delta} < +\infty$  for some  $\delta > 0$ , then there exists a tree-structured vector quantizer  $(\mathcal{Q}_t)$  such that

$$(4.1) \quad (\mathbb{E}\|x - \mathcal{Q}_t(x)\|^2)^{1/2} \leq C_+ 2^{-t/d},$$

where  $C_+$  is a positive constant depending only on the dimension  $d$  and the a priori density  $f(x)$ . Observe that the right-hand side of (4.1) differs from the right-hand side of (3.1) only by a constant independent of the quantizer's range size  $m = 2^t$ ; i.e., tree-structured quantizers are not suboptimal for their rate of convergence.

*Remark 2.* The upper bound (4.1) is easy to be obtained if  $\mathcal{X} = [0, 1]$ . Indeed, in this case, one can take  $\mathcal{S}$  to be the map which associates with  $x$  its binary expansion. We can apply this argument in case  $\mathcal{X}$  is a bounded subset of  $\mathbb{R}$ . In case  $\mathcal{X}$  is unbounded, tree-structured quantizers can be determined satisfying the upper bound

<sup>3</sup>See also Figure 6.1.

<sup>4</sup>In fact, it is tempting to conjecture that a tighter bound could possibly be proven for the exponent in the absence of feedback.

(4.1) (see Lemma 5.2 in [31]). The extension from the scalar to the vector case is straightforward.

Notice that if  $x', x'' \in \mathcal{X}$  are such that  $\mathcal{S}_t(x') = \mathcal{S}_t(x'')$ , then

$$(4.2) \quad \begin{aligned} \mathbb{E}\|x' - x''\|^2 &\leq \mathbb{E}(\|x' - \mathcal{Q}_t(x')\| + \|x'' - \mathcal{Q}_t(x'')\|)^2 \\ &\leq 2\mathbb{E}\|x' - \mathcal{Q}_t(x')\|^2 + 2\mathbb{E}\|x'' - \mathcal{Q}_t(x'')\|^2 \leq 2C_+^2 2^{-2t/d}. \end{aligned}$$

With a slight abuse of terminology, the map  $\mathcal{S}$  associated with a tree-structured vector quantizer will be called a dyadic expansion map. We now show how a transmission scheme can be built starting from  $\mathcal{S}$  and a family of its truncations' right inverses  $\mathcal{S}_t^{-1}$ .

Consider a sequence of integers  $m_1, m_2, \dots \in \mathbb{N}$  such that  $m_{t-1} \leq m_t$  for all  $t$  and a family of maps

$$(4.3) \quad \tilde{E}_t : \mathcal{Y}^{m_t} \rightarrow \mathcal{Y}, \quad \tilde{D}_t : \mathcal{Z}^t \rightarrow \mathcal{Y}^{m_t}.$$

We can define the map  $\tilde{\mathcal{E}} : \mathcal{Y}^{\mathbb{N}} \rightarrow \mathcal{Y}^{\mathbb{N}}$  by letting the value of  $\tilde{\mathcal{E}}((w_s)_{s=1}^{\infty})$  at time  $t$  be equal to  $\tilde{E}_t(w_1, \dots, w_{m_t})$ . We also put  $\tilde{\mathcal{E}}_t := \pi_t \circ \tilde{\mathcal{E}}$ . Notice that since  $\tilde{\mathcal{E}}_t((w_s)_{s=1}^{\infty})$  depends on  $w_1, \dots, w_{m_t}$  only, then  $\tilde{\mathcal{E}}_t$  is actually a map from  $\mathcal{Y}^{m_t}$  to  $\mathcal{Y}^t$ . Finally encoders and decoders are defined by  $\mathcal{E}_t := \tilde{\mathcal{E}}_t \circ \mathcal{S}_{m_t}$  and  $\mathcal{D}_t := \mathcal{S}_{m_t}^{-1} \circ \tilde{D}_t$ . The overall sequence of maps is described by the scheme

$$(4.4) \quad \begin{array}{ccccccccc} \mathcal{X} & \xrightarrow{\mathcal{S}_{m_t}} & \mathcal{Y}^{m_t} & \xrightarrow{\tilde{\mathcal{E}}_t} & \mathcal{Y}^t & \xrightarrow{\text{Channel}} & \mathcal{Z}^t & \xrightarrow{\tilde{D}_t} & \mathcal{Y}^{m_t} & \xrightarrow{\mathcal{S}_{m_t}^{-1}} & \mathcal{X} \\ x & \longmapsto & (w_s)_{s=1}^{m_t} & \longmapsto & (y_s)_{s=1}^t & \longmapsto & (z_s)_{s=1}^t & \longmapsto & (\hat{w}_s(t))_{s=1}^{m_t} & \longmapsto & \hat{x}_t. \end{array}$$

In other words, in this scheme we first use the dyadic expansion map to transform  $x$  into a string of bits  $(w_1, w_2, \dots, w_{m_t}, \dots)$ , and then we encode the latter into a sequence of channel inputs. The received data are decoded by a block decoder providing an estimated version  $(\hat{w}_1(t), \hat{w}_2(t), \dots, \hat{w}_{m_t}(t))$  of  $(w_1, w_2, \dots, w_{m_t})$  (whose components in general depend on  $t$ ) which is translated to an estimate  $\hat{x}_t$  of  $x$ .

**5. Low-complexity coding schemes.** In this section, trade-offs between computational complexity and performance of the coding schemes are investigated. First, in section 5.1, a simple linear-time encodable/decodable scheme is analyzed, showing that the estimation error converges to zero subexponentially fast with degree  $\alpha = 1/2$ . Then, in section 5.2, lower bounds on the estimation error are obtained: It is shown that encoding schemes with finite memory (i.e., that can be implemented by finite-state automata controlled by the dyadic expansion of the source vector) have estimation error bounded away from zero, while finite-window linear-time encodable schemes (i.e., such that each channel input can be written as a deterministic function of a finite number of bits of the dyadic expansion of the source vector) cannot achieve a convergence degree larger than  $1/2$ .

**5.1. A repetition coding scheme.** Let  $\mathcal{S}_t : \mathcal{X} \rightarrow \{0, 1\}^t$  be the truncated dyadic expansion map introduced in section 4, and let  $\mathcal{S}_t^{-1} : \{0, 1\}^t \rightarrow \mathcal{X}$  be one of its right inverses. If a coding scheme with  $\mathcal{E} = \mathcal{S}$  were simply used, i.e., if the bits of the dyadic expansion were directly sent through the channel, then the estimation error  $\Delta_t$  would not converge to 0 as  $t \rightarrow \infty$ . Indeed, with probability  $\varepsilon$  the first bit of  $\mathcal{S}(x)$  would be lost with no possibility of recovering it. It is therefore necessary to introduce redundancy in order to cope with channel erasures. The simplest way to

do that consists in using repetition schemes. Of course, since the different bits of the binary expansion  $\mathcal{S}(x)$  require different levels of protection, they need to be repeated with a frequency which is monotonically increasing in their significance.

The encoder we propose here is of the following type: at time  $t$ , the bit  $y_t$  to be sent through the channel coincides with  $w_{j_t}$ , the bit in position  $j_t$  of the dyadic expansion  $\mathcal{S}(x)$ . The encoder in this way will depend on the choice of  $j_t$  and fits in the scheme proposed in (4.3) simply by taking  $m_t := \max\{j_1, j_2, \dots, j_t\}$ .

In the scheme we propose,  $j_t$  is selected as follows. Fix a positive real  $q$  and define  $\tau_0 = 0$  and  $\tau_k = \lceil q \rceil + \lceil 2q \rceil + \dots + \lceil kq \rceil$  for  $k \in \mathbb{N}$ . Notice that, for any  $t \in \mathbb{N}$ , there exists a unique  $k$  such that  $\tau_{k-1} + 1 \leq t \leq \tau_k$ . Then define  $j_t := t - \tau_{k-1}$ . In other words, we have

$$(5.1) \quad (y_s)_{s=1}^\infty = \tilde{\mathcal{E}}((w_s)_{s=1}^\infty) = (w_1, w_2, \dots, w_{\lceil q \rceil}, w_1, w_2, \dots, w_{\lceil 2q \rceil}, w_1, w_2, \dots, w_{\lceil 3q \rceil}, \dots).$$

In any scheme of this kind the decoding is elementary. The output of the decoder  $(\hat{w}_j(t))_{j=1}^{m_t} \in \{0, 1\}^{m_t}$  may be given by

$$\hat{w}_j(t) = \begin{cases} z_s & \text{if } \exists s \leq t \text{ such that } j(s) = j \text{ and } z_s \neq ?, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that this decoding scheme has complexity growing linearly in  $t$ . Indeed, it admits the following natural recursive implementation. First, initialize  $\hat{w}_j(0) = 0$  for all  $j = 0$ . Then, for all  $t \geq 0$ , upon receiving  $z_{t+1}$  we compute  $(\hat{w}_j(t+1))_{j=1}^{m_{t+1}}$  as

$$(5.2) \quad \hat{w}_j(t+1) = \begin{cases} z_{t+1} & \text{if } j = j(t+1) \text{ and } z_{t+1} \neq ?, \\ \hat{w}_j(t) & \text{otherwise.} \end{cases}$$

PROPOSITION 5.1. Consider the repetition coding scheme defined by (5.1) and (5.2) on the BEC with erasure probability  $\varepsilon$ . Then the root mean squared error satisfies

$$(5.3) \quad \Delta_t \leq p(t)2^{-\beta t^{1/2}},$$

where

$$\begin{aligned} \beta = \frac{\sqrt{2q}}{d}, \quad p(t) = C_1 & \quad \text{if } q < \frac{d \log \varepsilon^{-1}}{2}, \\ \beta = \frac{\sqrt{2q}}{d}, \quad p(t) = C_2 \sqrt{t} & \quad \text{if } q = \frac{d \log \varepsilon^{-1}}{2}, \\ \beta = \frac{\log \varepsilon^{-1}}{\sqrt{2q}}, \quad p(t) = C_3 & \quad \text{if } q > \frac{d \log \varepsilon^{-1}}{2}, \end{aligned}$$

with  $C_1, C_2, C_3$  positive constants depending only on  $q, \varepsilon$ , and  $d$ .

Proof. Let us fix some  $t \in \mathbb{N}$ . Define  $v_j := |\{1 \leq \tau \leq t \mid j(\tau) = j\}|$  and observe that  $v_j = 0$  if  $j > m_t$ . For  $j = 0, 1, \dots, m_t$ , consider the events  $A_j := \{\hat{w}_1(t) = w_1, \dots, \hat{w}_j(t) = w_j\}$ ,  $B_j := A_j \setminus A_{j+1}$ . Notice that the events  $B_j$  are pairwise disjoint, and  $\mathbb{P}(\bigcup_{j=0}^{m_t-1} B_j \cup A_{m_t}) = 1$ . Moreover, observe that  $\mathbb{P}(B_j) \leq \mathbb{P}(\hat{w}_{j+1}(t) \neq w_{j+1}) \leq \varepsilon^{v_{j+1}}$ .

Notice that under the constraints posed by the event  $A_j^t$  we have that the first  $j$  bits of  $S(x)$  and of  $S(\hat{x}_t)$  coincide. Hence, by (4.2),  $\mathbb{E}[||x - \hat{x}_t||^2 | A_j^t] \leq 2C_+^2 2^{-2j/d}$ . Now for simplicity we assume that  $t = \tau_k$  for some  $k$ . In this case we have that  $m_t = \lceil kq \rceil$ . Moreover we have that  $v_j = k + 1 - \min\{h : \lceil qh \rceil \geq j\}$ . Observe now that since  $j \leq \lceil x \rceil$  if and only if  $j < x + 1$ , we can argue that

$$\min\{h : \lceil qh \rceil \geq j\} = \min\{h : qh + 1 > j\} = \min\{h : h > (j - 1)/q\} = \left\lfloor \frac{1}{q}(j - 1) \right\rfloor + 1.$$

This implies that, for  $j = 0, 1, \dots, \lceil kq \rceil$ , we have  $v_j = k - \lfloor \frac{j-1}{q} \rfloor$ . Therefore,

$$\begin{aligned} \Delta_t^2 &= \sum_{j=0}^{\lceil kq \rceil - 1} \mathbb{E}[||x - \hat{x}_t||^2 | B_j] \mathbb{P}(B_j) + \mathbb{E}[||x - \hat{x}_t||^2 | A_{\lceil kq \rceil}] \mathbb{P}(A_{\lceil kq \rceil}) \\ &\leq 2C_+^2 \left[ \sum_{j=0}^{\lceil kq \rceil - 1} 2^{-2j/d} \varepsilon^{v_{j+1}} + 2^{-2\lceil kq \rceil/d} \right] \\ &\leq 2C_+^2 \left[ \sum_{j=0}^{\lceil kq \rceil - 1} 2^{-2j/d} \varepsilon^{k - \lfloor j/q \rfloor} + 2^{-2\lceil kq \rceil/d} \right] \\ &\leq 2C_+^2 \left[ \sum_{j=0}^{\lceil kq \rceil - 1} 2^{-2j/d} \varepsilon^{k - j/q} + 2^{-2qk/d} \right]. \end{aligned}$$

Since  $t = \tau_k = \sum_{j=1}^k \lceil jq \rceil \leq \sum_{j=1}^k (jq + 1) = \frac{q}{2}k^2 + \frac{q+2}{2}k$ , we have  $k \geq \sqrt{\frac{2t}{q}} - \frac{q+2}{2q}$ . Then the claim easily follows from a straightforward computation.  $\square$

*Remark 3.* Notice that Proposition 5.1 implies that a convergence degree  $\alpha = 1/2$  is achievable for any choice of the positive parameter  $q$ , without any knowledge of the value of the erasure probability  $\varepsilon \in [0, 1]$ . If one knows  $\varepsilon$ , then it is possible to optimize the convergence rate  $\beta$  by choosing  $q = \frac{d \log \varepsilon^{-1}}{2}$ .

**5.2. A trade-off result between performance and complexity.** We shall now show how complexity limitations imply lower bounds to the error decay stronger than Theorem 3.1. In particular we shall prove that, for certain classes of encoders (finite-window and finite-state automata), exponential decay of error can never be achieved. As before, let us assume that  $\mathcal{S} : \mathcal{X} \rightarrow \{0, 1\}^{\mathbb{N}}$  is the dyadic expansion map introduced in section 4, and consider encoders  $\tilde{\mathcal{E}} : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$  of the form  $\tilde{\mathcal{E}}((w_s)_{s=1}^\infty)_t = \tilde{E}_t(w_1, \dots, w_{m_t})$ , for some finite integer  $m_t$ , and a map  $\tilde{E}_t : \mathcal{Y}^{m_t} \rightarrow \mathcal{Y}$ .

In general,  $\tilde{E}_t$  may actually depend on a proper subset of the  $m_t$  bits  $\{1, 2, \dots, m_t\}$ . Consider the minimal  $\Theta_t \subseteq \{1, 2, \dots, m_t\}$ , which allows one to write

$$\tilde{E}_t((w_s)_{s=1}^{m_t}) = f_t((w_s)_{s \in \Theta_t})$$

for a suitable function  $f_t : \Theta_t \rightarrow \{0, 1\}$ . Let  $n_t = |\Theta_t|$ . The encoder  $\tilde{\mathcal{E}}$  is called a finite-window if  $n_t$  is bounded in  $t$ . With each encoder it is possible to associate, for every  $j, t \in \mathbb{N}$ , the quantity  $\omega_j(t) := \sum_{1 \leq s \leq t} \mathbb{1}_{\Theta_s}(j)$ , counting the number of channel inputs up to time  $t$ , which have been affected by  $w_j$ . Define

$$\chi_t := \sum_{j \in \mathbb{N}} \omega_j(t) = \sum_{s \leq t} n_s.$$

The quantity  $\chi_t$  is related to the complexity of the encoder  $\tilde{\mathcal{E}}$ . If the maps  $f_t$  are  $\mathbb{Z}_2$ -linear and separately computed, then  $\chi_t$  provides an upper bound to the number of binary operations implemented by the encoder up to time  $t$ . However, there could be hidden recursive links among the  $f_t$  capable of lowering the real computational complexity. In any case, for brevity, we shall refer to  $\chi_t$  as the complexity function of the encoder. The following is our main result, relating the root mean squared error  $\Delta_t$  to the complexity function  $\chi_t$ .

**THEOREM 5.2.** *For any transmission scheme for the BEC, with erasure probability  $\varepsilon$ , consisting of an encoder with complexity function  $\chi_t$ , it holds that*

$$(5.4) \quad \Delta_t \geq C 2^{-\sqrt{\frac{1}{d}\chi_t \log \varepsilon^{-1}}},$$

where  $C > 0$  is a constant depending only on  $d$ , the erasure probability  $\varepsilon$ , and the density function  $f$  of the random vector  $x$ .

*Proof.* Assume that, at time  $t$ , all the  $\omega_j(t)$  channel inputs affected by the  $j$ th bit  $w_j$  have been erased. Then there is clearly no way for the decoder to reliably recover  $w_j$  from the channel output. This implies that  $\Delta_t^2 \geq C_1 \sup_{j \in \mathbb{N}} \{2^{-2j/d} \varepsilon^{\omega_j(t)}\}$  for some constant  $C_1 > 0$  depending only on  $d$  and  $f$ , independently from the way the decoders are chosen. It will prove convenient to consider the looser bounds

$$\Delta_t^2 \geq C_1 \sup \left\{ 2^{-2j} \varepsilon^{\omega_j(t)} : 1 \leq j \leq s \right\} \geq C_1 \psi_s(\omega_1(t), \dots, \omega_s(t)) \quad \forall s \in \mathbb{N},$$

where  $\psi_s(\omega_1, \dots, \omega_s) := \frac{1}{s} \sum_{j=1}^s 2^{-2j/d} \varepsilon^{\omega_j}$ . Hence, for every possible  $s$ ,

$$(5.5) \quad \Delta_t^2 \geq C_1 \inf \{ \psi_s(\omega_1, \dots, \omega_s) : \omega \in M_s \},$$

where  $M_s := \{ \omega_1, \dots, \omega_s \in (\mathbb{R}^+)^s : \sum_j \omega_j = \chi_t \}$ . Since the function  $\psi_s$  is strictly convex, it admits a unique minimum on the convex compact set  $M_s$ . Using Lagrange multipliers, the unique stationary point  $(\omega_j^*)$  of  $\psi_s(\omega_1, \dots, \omega_s)$  on the hyperplane  $M_s$  has to satisfy, for all  $j \leq s$ ,  $\omega_j^* = \varsigma - \rho j$ , where  $\rho := \frac{\ln 4}{d \ln \varepsilon^{-1}} = \frac{2}{d \log \varepsilon^{-1}} > 0$ , and  $\varsigma = \frac{\chi_t}{s} + \rho \frac{s+1}{2}$ . We have that  $\omega^* \in M_s$  if and only if  $\omega_s^* \geq 0$ , which is equivalent to  $s \leq \frac{1}{2}(1 + \sqrt{1 + (8\chi_t/\rho)})$ . A possible choice is provided by  $s^* = \lfloor \sqrt{2\chi_t/\rho} \rfloor$ . We thus obtain

$$(5.6) \quad \Delta_t^2 \geq C_1 \inf_{\omega \in M_{s^*}} \psi_{s^*}(\omega_1, \dots, \omega_{s^*}) = \psi_{s^*}(\omega_1^*, \dots, \omega_{s^*}^*) = C_1 e^{-\varsigma \ln \varepsilon^{-1}}.$$

We can estimate  $\varsigma^*$  as

$$\begin{aligned} \varsigma^* &= \frac{\chi_t}{\lfloor \sqrt{\frac{2\chi_t}{\rho}} \rfloor} + \rho \frac{\lfloor \sqrt{\frac{2\chi_t}{\rho}} \rfloor + 1}{2} \leq \frac{\chi_t}{\sqrt{\frac{2\chi_t}{\rho}} - 1} + \frac{\rho}{2} \left( \sqrt{\frac{2\chi_t}{\rho}} + 1 \right) \\ &= \sqrt{\rho} \frac{2\chi_t - \rho/2}{\sqrt{2\chi_t} - \sqrt{\rho}} \leq \rho \left( \sqrt{\frac{2\chi_t}{\rho}} + \frac{2\sqrt{2}-1}{2\sqrt{2}-2} \right), \end{aligned}$$

the last equality following from the assumption  $\chi_t \geq \rho$ . Inserting this last estimation inside (5.6), the claim follows.  $\square$

We have the following straightforward consequence for finite-window encoders which show that the degree  $\alpha = 1/2$  cannot be beaten.

**COROLLARY 5.3.** *For any transmission scheme for the BEC, with erasure probability  $\varepsilon$ , consisting of a finite-window encoder with  $n_t \leq n_{\max}$  for every  $t$ , it holds that*

$$(5.7) \quad \Delta_t \geq C 2^{-\beta t^{1/2}},$$

where  $\beta = \sqrt{(n_{\max} \log \varepsilon^{-1})/d}$  and where  $C > 0$  is a constant depending only on  $d$ , the erasure probability  $\varepsilon$ , and the density function  $f$  of the random vector  $x$ .

*Remark 4.* In the case of the repetition encoders treated in section 5.1, we have that  $n_{\max} = 1$ . If we compare (5.7) with (5.3), we have thus established that among the repetition schemes ( $n_{\max} = 1$ ), the example treated in section 5.1 is optimal from the point of view of the asymptotic performance (both degree and rate of convergence).

The bound (5.4) implies that, in order to obtain exponential convergence of the error,  $\chi_t$  needs to grow at least quadratically in  $t$  or, equivalently, that  $\frac{1}{t}\chi_t$ , i.e., the average number of bits of the dyadic expansion  $\mathcal{S}(x)$  the channel inputs depend on, grows at least linearly in  $t$ . Indeed, as we shall see, the random linear codes proposed in section 6 have exactly this property. However, observe that this does not imply that linear-time encodable schemes cannot attain exponential error decays in any case, since  $\chi_t$  is, as already noticed, only an upper bound to the complexity of the encoder, intended as the minimum number of operations required by any implementation of the encoder. A possibility would be to consider maps  $f_t$  which, despite not being finite-window, can still be computed with bounded complexity in some recursive way. The most obvious choice would be to consider finite-state automata schemes. Unfortunately, such schemes yield very poor performance, as will be shown in the next subsection. A less simple choice (and which will not be pursued here) would be to consider encoders obtained as serial concatenations of finite-window with finite-state automata schemes.

**5.2.1. Finite-state automata encoders.** Encoders which can be implemented as finite-state automata yield very poor performance. In fact, the root mean squared error  $\Delta_t$  in this case does not converge to 0 as  $t \rightarrow +\infty$ . Indeed, assume we are given a finite-state alphabet  $A$  and two maps  $\xi : A \times \{0, 1\} \rightarrow A$ ,  $\rho : A \times \{0, 1\} \rightarrow \{0, 1\}$ . Moreover, fix an initial state  $a^* \in A$ . To the quadruple  $(A, \xi, \rho, a^*)$  we can naturally associate an encoder  $\tilde{\mathcal{E}}$  as follows. Given  $(w_s)_{s=1}^\infty \in \{0, 1\}^\mathbb{N}$ , recursively define  $(y_s)_{s=1}^\infty = \tilde{\mathcal{E}}((w_s)_{s=1}^\infty)$  by

$$\begin{cases} a_{t+1} = \xi(a_t, w_t), & a_0 = a^*, \\ y_t = \rho(a_t, w_t). \end{cases}$$

Notice that the state updating map  $\xi$  together with the initial condition  $a_0 = a^*$  yields a sequence of maps  $\xi^{(t)} : \{0, 1\}^t \rightarrow A$  such that  $a_{t+1} = \xi^{(t)}(w_1, \dots, w_t)$ . If we choose  $t = t_0$  in such a way that  $2^{t_0} > |A|$ , the map  $\xi^{(t_0)}$  is necessarily not injective. Hence, there exist two different input truncated sequences  $(w'_1, \dots, w'_{t_0})$  and  $(w''_1, \dots, w''_{t_0})$  such that  $\xi^{(t_0)}(w'_1, \dots, w'_{t_0}) = \xi^{(t_0)}(w''_1, \dots, w''_{t_0})$ . Consider the event  $A = \{w_k = w'_k, z_k = ? \text{ for } k = 1, \dots, t_0\}$ . Clearly, conditioned on  $A$ , the decoder, for any  $t \geq t_0$ , will decode incorrectly at least one information bit in the first  $t_0$  position with positive probability independent from  $t$ . Hence,  $\Delta_t^2 \geq \mathbb{E} [\|x - \hat{x}_t\|^2 | A] \mathbb{P}(A) \geq 2^{-2t_0/d} \mathbb{P}(A)$ .

**6. A coding scheme with exponential error rates.** The goal of this section is to show that, removing the complexity bounds, exponential convergence can be achieved. The proposed scheme will require quadratic computational complexity at the encoder and cubic complexity at the decoder.

We shall use random coding arguments employing linear tree codes over the binary field  $\mathbb{Z}_2$ . These arguments were first developed in the context of convolutional codes [43, 44, 16] and recently applied in the framework of anytime information theory [36, 37]. For the reader's convenience, and since those results have not appeared anywhere else in this form, we shall present self-contained proofs. The coding strategy

we shall propose is very close in spirit to those in [36, Thm. 5.1] and [37, Thm. 5.1], the main difference being that we use linear convolutional codes instead of general random convolutional codes. Our choice has the double advantage of lowering the memory and complexity requirements for the encoder and the decoder (see section 6.3) and improving the achievable error rate for a significant range of values of  $\varepsilon$  (see Theorem 6.3 and Remark 5).

**6.1. A random causal linear coding scheme.** In this section we shall identify the binary set  $\mathcal{Y} = \{0, 1\}$  with the binary field  $\mathbb{Z}_2$  of the integers modulo 2.

Fix a rate  $0 < R < 1$  and for any  $t$  let  $m_t := \lfloor Rt \rfloor$ . Consider a random, doubly infinite, binary matrix  $\phi \in \mathbb{Z}_2^{\mathbb{N} \times \mathbb{N}}$  distributed as follows:  $\phi_{ij} = 0$  for all  $j > Ri$  (i.e., for all  $j \geq m_i + 1$ ), while  $\{\phi_{ij}\}_{1 \leq j \leq Ri}$  is a family of mutually independent random variables with identical uniform distribution over  $\mathbb{Z}_2$ . As is customary in random coding arguments, we shall assume the random matrix  $\phi$  to be independent from the source vector  $x$  as well as from the channel, and known a priori both at the transmitting and receiving ends. Let us naturally identify the random matrix  $\phi$  with the corresponding random  $\mathbb{Z}_2$ -linear operator  $\tilde{\mathcal{E}} : \mathbb{Z}_2^{\mathbb{N}} \rightarrow \mathbb{Z}_2^{\mathbb{N}}$ . Consider the truncated encoder

$$(6.1) \quad \tilde{\mathcal{E}}_t : \mathbb{Z}_2^{m_t} \rightarrow \mathbb{Z}_2^t, \quad \tilde{\mathcal{E}}_t((w_s)_{s=1}^{m_t}) := \pi_t(\phi \mathbf{w}),$$

where  $\mathbf{w} \in \mathbb{Z}_2^{\mathbb{N}}$  is such that  $\pi_{m_t} \mathbf{w} = (w_s)_{s=1}^{m_t}$ . Now, let  $\mathcal{S} : \mathcal{X} \rightarrow \mathbb{Z}_2^{\mathbb{N}}$  be the dyadic expansion map introduced in section 4, and define the encoding scheme  $\mathcal{E} : \mathcal{X} \rightarrow \mathbb{Z}_2^{\mathbb{N}}$  as the composition  $\mathcal{E} = \tilde{\mathcal{E}} \circ \mathcal{S}$ .

For the decoding part, we shall consider maximum a posteriori decoders  $\tilde{\mathcal{D}}_t$ . For the special case of the BEC, given the channel outputs  $z_t$ , the decoded block at time  $t$ ,  $(\hat{w}_s(t))_{s=1}^{m_t} = \tilde{\mathcal{D}}_t((z_s)_{s=1}^t)$ , is defined to be any vector in  $\{0, 1\}^{m_t}$  which is compatible with the observed channel output  $(z_s)_{s=1}^t$ . Formally, let  $\Xi_t := \{s \in \{1, \dots, t\} : z_s \neq ?\}$  be the set of nonerased positions up to time  $t$ , and let  $\pi_{\Xi_t} : \mathbb{Z}_2^t \rightarrow \mathbb{Z}_2^{\Xi_t}$  be the canonical projection. Then a maximum a posteriori decoder  $\mathcal{D}_t : \{0, 1, ?\}^t \rightarrow \mathbb{Z}_2^{m_t}$  maps the channel output  $(z_s)_{s=1}^t$  into any binary string  $(\hat{w}_s(t))_{s=1}^{m_t}$  such that

$$(6.2) \quad \pi_{\Xi_t} \tilde{\mathcal{E}}_t((\hat{w}_s(t))_{s=1}^{m_t}) = \pi_{\Xi_t}(z_s)_{s=1}^t = \pi_{\Xi_t} \tilde{\mathcal{E}}_t((w_s)_{s=1}^{m_t}).$$

Finally, the overall decoder is defined as the composition  $\mathcal{D}_t := \mathcal{S}_{m_t}^{-1} \circ \tilde{\mathcal{D}}_t$ .

**6.2. Performance analysis.** We now analyze the coding scheme we have introduced. Notice first of all that the decoded block  $(\hat{w}_s(t))_{s=1}^{m_t} = \tilde{\mathcal{D}}_t((z_s)_{s=1}^t) \in \mathbb{Z}_2^{m_t}$  is uniquely defined, and correct, whenever the linear map  $\pi_{\Xi_t} \tilde{\mathcal{E}}_t : \mathbb{Z}_2^{m_t} \rightarrow \mathbb{Z}_2^{\Xi_t}$  is injective. However, our analysis requires more detailed information regarding the location of the incorrectly decoded information bits when injectivity is lost. To this end, let  $\{\delta_1, \delta_2, \dots, \delta_{m_t}\}$  be the canonical basis of  $\mathbb{Z}_2^{m_t}$  and, for  $0 \leq j \leq m_t$ , consider the subspace<sup>5</sup>  $K_j := \text{span}(\delta_{j+1}, \dots, \delta_{m_t}) \subseteq \mathbb{Z}_2^{m_t}$ . For  $0 \leq j \leq m_t$ , define the event  $A_j := \{\ker(\pi_{\Xi_t} \tilde{\mathcal{E}}_t) \subseteq K_j\}$ . Also, let us define  $B_j := A_{j-1} \setminus A_j$  for  $1 \leq j \leq m_t$ . Observe that  $A_j \subseteq A_{j-1}$  and that  $A_0$  coincides with the whole sample space  $\Omega$ . Hence, for every  $t \in \mathbb{N}$ , the sample space admits the partition

$$(6.3) \quad \Omega = \bigcup_{1 \leq j \leq m_t} B_j \bigcup A_{m_t}.$$

<sup>5</sup>We shall use the standard convention  $\text{span}(\emptyset) := \{0\}$ .

Notice now that we can deduce from (6.2) that  $(w_s - \hat{w}_s(t))_{s=1}^{m_t} \in \ker \pi_{\Xi_t} \tilde{\mathcal{E}}_t$ . Therefore, if  $A_j$  holds true, then  $(\hat{w}_s(t))_{s=1}^j = (w_s)_{s=1}^j$ ; i.e., the first  $j$  bits of the quantization of  $x$  are correctly decoded. We immediately get from (4.2) that if  $A_j$  occurs, then

$$(6.4) \quad \|\hat{x}_t - x\|^2 \leq 4d2^{-2j/d}, \quad 0 \leq j \leq m_t.$$

The following result characterizes the average mean squared error of the random coding scheme  $(\mathcal{E}, \mathcal{D})$  over the BEC. Here the average has to be considered with respect to the randomness of the vector  $x$ , the channel, as well as the matrix  $\phi$ . For  $\varepsilon \in [0, 1]$  and  $d \in \mathbb{N}$ , define

$$(6.5) \quad \underline{\beta}'(d, \varepsilon, R) := \min \left\{ \frac{1}{d}R, \frac{1}{2} \min_{0 \leq \eta \leq 1} D(\eta || 1 - \varepsilon) + \lfloor \eta - R \rfloor_+ \right\},$$

where  $D(x || y) := x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y}$  denotes the binary Kullback–Leiber distance and where  $\lfloor x \rfloor_+ := \max\{0, x\}$ .

**THEOREM 6.1.** *Assume transmission over the BEC. Then, for all  $0 < R < 1$ , the average estimation error of the above-described random coding scheme satisfies*

$$(6.6) \quad (\mathbb{E} \|x - \hat{x}_t\|^2)^{1/2} \leq C \sqrt{t} 2^{-\beta'(d, \varepsilon, R) t}$$

for all  $t \in \mathbb{N}$ , where  $C > 0$  is a constant depending only on  $d$ ,  $R$ , and  $\varepsilon$ .

*Proof.* Using (6.3) and (6.4), we obtain

$$(6.7) \quad \begin{aligned} \mathbb{E} [\|\hat{x}_t - x\|^2] &= \sum_{j=1}^{m_t} \mathbb{E} [\|\hat{x}_t - x\|^2 | B_j] \mathbb{P}(B_j) + \mathbb{E} [\|\hat{x}_t - x\|^2 | A_{m_t}] \mathbb{P}(A_{m_t}) \\ &\leq \sum_{j=1}^{m_t} \mathbb{P}(B_j) 4d2^{-2(j-1)/d} + 4d2^{-2m_t/d}. \end{aligned}$$

In order to estimate  $\mathbb{P}(B_j)$ , first we claim that the event  $B_j$  implies that the column  $\pi_{\Xi_t} \tilde{\mathcal{E}}_t \delta_j$  belongs to the subspace  $\pi_{\Xi_t} \tilde{\mathcal{E}}_t K_j$ . Indeed,  $\overline{A_j}$  implies that there exists some  $v \in \mathbb{Z}_2^{m_t}$  such that  $\pi_{\Xi_t} \tilde{\mathcal{E}}_t v = 0$ , and  $v_i \neq 0$  for some  $i \geq j$ . On the other hand,  $A_{j-1}$  implies that such a  $v$  has  $v_i = 0$  for all  $i < j$ . Hence, if we define  $v' \in \mathbb{Z}_2^{m_t}$  by  $v'_i = 0$  for  $i \leq j$ , and  $v'_i = v_i$  for  $i > j$ , we have that  $v' \in K_j$  and  $0 = \pi_{\Xi_t} \tilde{\mathcal{E}}_t v = \pi_{\Xi_t} \tilde{\mathcal{E}}_t \delta_j + \pi_{\Xi_t} \tilde{\mathcal{E}}_t v'$ . Therefore  $\pi_{\Xi_t} \tilde{\mathcal{E}}_t \delta_j = \pi_{\Xi_t} \tilde{\mathcal{E}}_t v' \in \pi_{\Xi_t} \tilde{\mathcal{E}}_t K_j$ , as claimed.

Observe that  $\tilde{\mathcal{E}}_t \delta_j$  is uniformly distributed over  $H_j := \text{span}(\delta_{\lceil j/R \rceil}, \dots, \delta_t) \subseteq \mathbb{Z}_2^t$ , and independent from  $\lambda_{\lceil j/R \rceil}^t$  (the latter being defined in (3.2)). It follows that  $\pi_{\Xi_t} \tilde{\mathcal{E}}_t \delta_j$  takes any value in  $\pi_{\Xi_t} H_j$  with probability  $2^{-\lambda_{\lceil j/R \rceil}^t}$ . Since  $|\pi_{\Xi_t} \tilde{\mathcal{E}}_t K_j| \leq |K_j| = 2^{m_t - j}$ , we have that, for every  $k = 0, \dots, t - \lceil j/R \rceil + 1$ ,

$$\begin{aligned} \mathbb{P}(B_j | \lambda_{\lceil j/R \rceil}^t = k) &\leq \mathbb{P}(\pi_{\Xi_t} \tilde{\mathcal{E}}_t \delta_j \in \pi_{\Xi_t} \tilde{\mathcal{E}}_t K_j | \lambda_{\lceil j/R \rceil}^t = k) \\ &\leq \min\{1, |K_j| 2^{-k}\} \\ &= 2^{-\lfloor j+k-m_t \rfloor_+}. \end{aligned}$$

From (3.3) it follows that

$$\begin{aligned}
 \mathbb{P}(B_j) &= \sum_{k=0}^{t-\lceil j/R \rceil + 1} P(B_j | \lambda_{\lceil j/R \rceil}^t = k) \mathbb{P}(\lambda_{\lceil j/R \rceil}^t = k) \\
 (6.8) \quad &\leq \sum_{k=0}^{t-\lceil j/R \rceil + 1} 2^{-\lfloor j+k-m_t \rfloor +} \binom{t-\lceil j/R \rceil + 1}{k} \varepsilon^{t-\lceil j/R \rceil + 1 - k} (1-\varepsilon)^k \\
 &\leq 2 \sum_{k=0}^{t-\lceil j/R \rceil + 1} 2^{-\lfloor j+k-m_t+1 \rfloor +} 2^{-(t-\lceil j/R \rceil + 1)D(\frac{k}{t-\lceil j/R \rceil + 1} || 1-\varepsilon)} \\
 &\leq 2t 2^{-(t-j/R) \min_{0 \leq \eta \leq 1} D(\eta || 1-\varepsilon) + \lfloor \eta - R \rfloor +},
 \end{aligned}$$

where the second inequality follows from standard estimations of the binomial coefficient (see, e.g., [12]), and the last one by setting  $\eta = \frac{k}{t-\lceil j/R \rceil + 1}$ , and the estimates  $R \leq 1, m_t \leq Rt, j/R \leq \lceil j/R \rceil \leq j/R + 1$ . Finally, (6.6) follows by substituting (6.8) into (6.7).  $\square$

Standard probabilistic arguments allow one to prove the following corollary of Theorem 6.1, characterizing the exponential error rate of a typical realization of the random coding scheme  $(\mathcal{E}, \mathcal{D})$ . Observe that the root mean squared error of the coding scheme is given by  $(\mathbb{E}[|\hat{x}_t - x|^2 | \phi])^{1/2}$ , which is a function of  $\phi$ , and hence a random variable.

**COROLLARY 6.2.** *Assume transmission over the BEC with erasure probability  $\varepsilon$ . Then, for all  $0 < R < 1$ , with probability one,*

$$(6.9) \quad (\mathbb{E}[|x - \hat{x}_t|^2 | \phi])^{1/2} \leq Ct^{3/2} 2^{-\underline{\beta}'(d, \varepsilon, R)t}$$

for a positive constant  $C$ .

*Proof.* For  $t \in \mathbb{N}$ , consider the event  $A_t := \{\mathbb{E}[|x - \hat{x}_t|^2 | \phi] \geq t^3 2^{-2t(\underline{\beta}'(d, \varepsilon, R))}\}$ . From Markov's inequality and Theorem 6.1, it follows that

$$\mathbb{P}[A_t] \leq t^{-3} 2^{2t\underline{\beta}'(d, \varepsilon, R)} \mathbb{E}[|x - \hat{x}_t|^2] \leq Ct^{-2},$$

so that the series  $\sum_t \mathbb{P}[A_t]$  is convergent and the Borel–Cantelli lemma implies that, with probability one,  $A_t$  occurs only for finitely many values of  $t \in \mathbb{N}$ . Then the claim easily follows.  $\square$

It is possible to derive another lower bound on the typical-case exponential error rate achieved by the random scheme  $(\mathcal{E}, \mathcal{D})$ , which turns out to be tighter than that provided by Corollary 6.2 for certain values of  $R$  and  $\varepsilon$ . For every  $0 \leq R \leq 1$ , define  $\gamma(R) := \min\{x \in [0, 1] : H(x) \geq 1 - R\}$ , and

$$\underline{\beta}''(d, \varepsilon, R) := \min \left\{ \frac{1}{d}R, \frac{1}{2} \min_{\gamma(R) \leq \eta \leq 1} \{H(\eta) - 1 + R - \eta \log \varepsilon\} \right\}.$$

The following result is proved in Appendix A.

**THEOREM 6.3.** *Assume transmission over the BEC with erasure probability  $\varepsilon$ . Then, for all  $0 < R < 1, \delta > 0$ , with probability one,*

$$(6.10) \quad (\mathbb{E}[|x - \hat{x}_t|^2 | \phi])^{1/2} \leq Kt 2^{-(\underline{\beta}''(d, \varepsilon, R) + \delta)t}$$

for a constant  $K > 0$ .

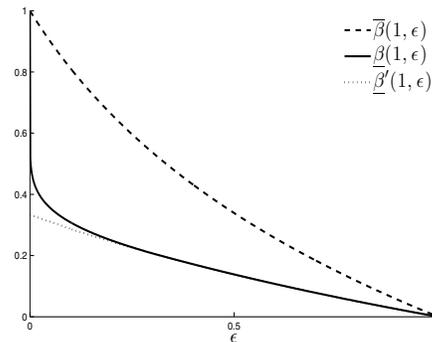


FIG. 6.1. Upper and lower bounds to the achievable estimation error exponent achievable on the BEC (as defined in (3.5), (6.11), and Remark 5, respectively) are plotted as a function of the erasure probability  $\varepsilon$  for  $d = 1$ .

*Remark 5.* It follows from Corollary 6.2 and Theorem 6.3 that, for all  $R < 1 - \varepsilon$ , random causal linear codes achieve exponential convergence rate. Optimizing over  $R \in ]0, 1 - \varepsilon[$ , this shows that the exponent

$$(6.11) \quad \underline{\beta}(d, \varepsilon) := \max_{0 \leq R \leq 1} \max\{\underline{\beta}'(d, \varepsilon, R), \underline{\beta}''(d, \varepsilon, R)\}$$

is achievable. In Figure 6.1 the upper and lower bounds to the error exponent, i.e.,  $\bar{\beta}(d, \varepsilon)$  and  $\underline{\beta}(d, \varepsilon)$ , are plotted as functions of the erasure probability  $\varepsilon$ , in the case  $d = 1$ . Define  $\underline{\beta}'(d, \varepsilon) := \max\{\underline{\beta}'(d, \varepsilon, R) : R \in [0, 1]\}$  and  $\underline{\beta}''(d, \varepsilon) := \max\{\underline{\beta}''(d, \varepsilon, R) : R \in [0, 1]\}$ . Then it is not difficult to see that  $\lim_{\varepsilon \downarrow 0} \underline{\beta}'(d, \varepsilon) = 1/(d + 2)$ , while  $\lim_{\varepsilon \downarrow 0} \underline{\beta}''(d, \varepsilon) = 1/d$ . Hence, Theorem 6.3 becomes particularly relevant for small erasure probabilities, showing that the noiseless error exponent  $1/d$  (see section 4) is recovered in the limit of vanishing noise: this does not follow from the average-code analysis of Theorem 3.1.

**6.3. Computational complexity of the scheme.** Observe that the number  $n_t$  of binary operations required in order to compute the channel input  $y_t = \tilde{\mathcal{E}}_t((w_s)_{s=1}^{m_t})$  equals the number of nonzero entries of the  $t$ th row of the infinite random matrix  $\phi$ . By the way  $\phi$  has been defined,  $n_t$  is a binomial random variable of parameters  $m_t$  and  $1/2$ . Hence, the number of binary operations required by the encoder up to time  $t$ ,  $\chi_t := \sum_{s \leq t} n_s$ , has binomial distribution of parameters  $\frac{1}{2}m_t(m_t + 1)$  and  $1/2$ . Therefore, the worst-case encoding complexity (worst case with respect to the realization of  $\phi$ ) grows like  $\frac{1}{2}R^2t^2$ , while the strong law of large numbers implies that the typical encoder complexity  $\chi_t$  is such that  $\chi_t/\frac{1}{4}R^2t^2$  converges to 1 with probability one. Thus, the encoder complexity (both worst case and typical case) is quadratic in  $t$ . Further, observe that the memory requirements of the encoder are quadratic in  $t$ , for it is necessary to store  $m_t t$  binary values in order to memorize the finite truncation  $\mathcal{E}_t$  of the encoder  $\mathcal{E}$ .

In order to evaluate the decoder's computational complexity, observe that  $\tilde{\mathcal{D}}_t$  is required to solve the  $\mathbb{Z}_2$ -linear system

$$(6.12) \quad \pi_{\Xi_t} \tilde{\mathcal{E}}_t((w_s)_{s=1}^{m_t}) = \pi_{\Xi_t} (z_s)_{s=1}^t$$

at each time step  $t$ . This can be performed using Gaussian elimination techniques in order to reduce the matrix  $\pi_{\Xi_t} \tilde{\mathcal{E}}_t$  to a lower diagonal form. Notice that a sequential

implementation is possible; i.e., the part of  $\pi_{\Xi_t} \tilde{\mathcal{E}}_t$  which has been reduced in lower triangular form at time  $t$  does not require further processing in future times  $s > t$ . Since Gaussian elimination techniques require  $O(t^3)$  operations, we can conclude that the decoder complexity is at most  $O(t^3)$ . On the other hand, it might be possible to find algorithms for solving a linear system like (6.12) with number of operations  $o(t^3)$ ; see [41, pp. 247–248] for the analogous problem for linear systems over the reals. However, the system (6.12) cannot be solved using fewer operations than those required to verify that a given string  $v \in \mathbb{Z}_2^{m_t}$  is a solution. Using arguments similar to those outlined above, it is possible to show that, with probability one, this requires  $\Theta(t^2)$  binary operations. In summary, the complexity of maximum a posteriori decoding of linear convolutional codes on the BEC is at most  $O(t^3)$  and at least  $\Theta(t^2)$ .

**7. Simulation results for finite-window coding schemes.** We shall now present Monte Carlo simulation results for some finite-window  $\mathbb{Z}_2$ -linear coding schemes with low-complexity iterative decoding. These schemes are based on ideas similar to those of digital fountain codes (see [22], [28, Ch. 50]). The latter are widely used in many applications, such as data storage, or reliable transmission on broadcast channels with erasures. The main additional challenge posed by our application consists in providing unequal error protection to the source bits.

We propose the following random construction for finite-window encoders fitting in the framework of section 5. As usual, assume that we have a dyadic expansion  $\mathcal{S}$  mapping the vector  $x$  into an infinite string of bits  $(w_s)_{s=1}^\infty$ . We imagine that at each time  $t$  the encoder produces a bit  $y_t$  which is the (modulo-2) sum of a random number of randomly chosen  $w_s$ , namely,

$$y_t = \sum_{s \in \Theta_t} w_s,$$

where  $\Theta_t$  is a random subset of  $\mathbb{N}$ . We assume that the cardinality of  $\Theta_t$  is bounded, i.e.,  $|\Theta_t| \leq n_{\max}$ .

More precisely, fix  $n_{\max} \in \mathbb{N}$  and a probability distribution  $\mu(\cdot)$  on  $\{1, \dots, n_{\max}\}$ . Randomly generate a sequence  $(n_t)_{t \in \mathbb{N}}$  of independent random variables distributed according to  $\mu(\cdot)$ . Let  $(\nu_t(\cdot))_{t \in \mathbb{N}}$  be a sequence of probability distributions over  $\mathbb{N}$ , with  $\nu_t(\cdot)$  possibly depending on  $(n_s)_{s \leq t}$ . Then, for every  $t \geq 1$ , consider the random set  $\Theta_t := \{\theta_{1,t}, \theta_{2,t}, \dots, \theta_{n_t,t}\}$ , where  $\theta_{i,t}$  are independent random variables uniformly distributed according to  $\nu_t(\cdot)$ . Notice that in this way we have that  $|\Theta_t| \leq n_t \leq n_{\max}$ , and so the encoder complexity is linear in  $t$ .

For the decoding, a sequential implementation of the peeling algorithm is used, this being the standard decoding technique for digital fountain codes [22], [28, Ch. 50]. Such an algorithm works on an iteratively updated infinite hypergraph<sup>6</sup>  $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{H}_t)$  as explained below. At  $t = 0$ ,  $\mathcal{G}_0$  is initialized with vertex set  $\mathcal{V}_0 = \mathbb{N}$  and empty hyperedge set  $\mathcal{H}_0 = \emptyset$ . The estimates  $(\hat{w}_s(0))_{s \in \mathbb{N}}$  of the dyadic expansion  $\mathcal{S}(x)$  are in turn initialized arbitrarily in  $\{0, 1\}^\mathbb{N}$ . At each time  $t \geq 1$ , first update  $\mathcal{V}_t = \mathcal{V}_{t-1}$ ,  $\mathcal{H}_t = \mathcal{H}_{t-1}$ , and  $\hat{w}_s(t) = \hat{w}_s(t+1)$  for all  $s \in \mathbb{N}$ . Then

- if  $z_t = ?$ , then quit; if  $z_t \neq ?$ , update  $\mathcal{H}_t = \mathcal{H}_t \cup \{B_t\}$ , where  $B_t := \Theta_t \cap \mathcal{V}_t$ ;
- if  $|B_t| > 1$ , then quit; otherwise if  $B_t = \{v\}$  for some  $v \in \mathcal{V}_t$ , set  $\hat{w}_v(t) = z_t + \sum_{j \in \Theta_t \setminus \{v\}} \hat{w}_j(t)$ , and eliminate  $v$  from  $\mathcal{V}_t$  as well as from all the hyperedges  $h \in \mathcal{H}_t$  containing it;

<sup>6</sup>The term hypergraph [3, p. 7] refers to a pair  $(\mathcal{V}, \mathcal{H})$ , where  $\mathcal{V}$  is a discrete set and  $\mathcal{H}$  is a subset of  $\mathcal{P}(\mathcal{V})$ , the power set of  $\mathcal{V}$ .

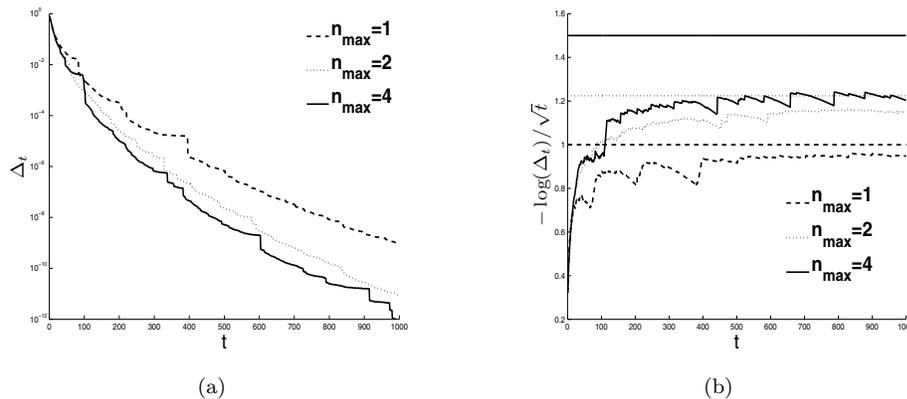


FIG. 7.1. Monte Carlo simulations of finite-window coding schemes on the BEC, with erasure probability  $\varepsilon = 0.5$ . The performance of three coding schemes is compared: these schemes were randomly generated accordingly to (7.1) and (7.2) with  $n_{\max} = 1, 2, 4$ , respectively. In (a) the root mean squared error  $\Delta_t$  is plotted as a function of the time  $t$  in log-linear scale. In (b)  $-\frac{1}{\sqrt{t}} \log \Delta_t$  is plotted as a function of  $t$ , together with the corresponding upper bounds  $\sqrt{\chi_t} \log \varepsilon^{-1}$  provided by Theorem 5.2. The number of samples used is 200000.

- if  $|h| \neq 1$  for all  $h \in \mathcal{H}_t$ , quit; otherwise, if there is some  $h = \{v\} \in \mathcal{H}_t$ , repeat the previous step.

The above-described algorithm requires an order of  $\chi_t = \sum_{s \leq t} n_s$  operations up to time  $t$ , and hence it has linear complexity in  $t$ . It is suboptimal with respect to the maximum a posteriori decoding: it may fail to correctly estimate the first  $j$  bits of the dyadic expansion  $\mathcal{S}(x)$  even when that would be possible using the maximum a posteriori decoder.

In Figure 7.1 we report Monte Carlo simulations of three finite-window encoding schemes, with  $n_{\max} = 1, 2, 4$ , respectively. The degree distribution  $\mu(\cdot)$  was chosen to be the truncated soliton one [28, p. 592]:

$$(7.1) \quad \mu(1) := \frac{1}{n_{\max}}, \quad \mu(n) := \frac{1}{n(n-1)} \quad \forall 2 \leq n \leq n_{\max}.$$

The distributions  $\nu_t$  have been selected as follows. We define  $\rho := 2(d \log \varepsilon^{-1})^{-1}$ ,  $s_t := \lfloor \sqrt{2\chi_t \rho^{-1}} \rfloor$ , and  $\varsigma_t = \frac{\chi_t}{s_t} + \rho \frac{s_t + 1}{2}$ , where  $\chi_t = \sum_{s \leq t} n_s$ . Then choose

$$(7.2) \quad \nu_t(j) := \begin{cases} \eta(\varsigma_t - \rho j) & \text{if } j \leq s_t, \\ 0 & \text{if } j > s_t. \end{cases}$$

Our choice was suggested by the optimization problem on the right-hand side of (5.5).

It is clear from Figure 7.1(a) that the three schemes have subexponential error decay and that increasing the degree allows one to obtain better convergence rates. Figure 7.1(b) shows that the convergence degree is  $\alpha = 1/2$ , as expected from the theory, while it is possible to recognize the different values of  $\beta$  of the three schemes in the asymptotic limit of  $-\frac{1}{\sqrt{t}} \log \Delta_t$ .

It should be emphasized that the choices of the distributions  $\mu$  and  $\nu_t$  were not optimized, but rather suggested by the literature on digital fountain codes and by Theorem 5.2, respectively. A theoretical analysis of the behavior of finite-window schemes, hopefully providing hints on the design of  $\mu$  and  $\nu_t$ , is left as a topic for future research.

**8. Conclusion and extensions.** The problem of anytime reliable transmission of a real-valued random vector through a digital noisy channel has been addressed. Upper and lower bounds on the highest exponential rate achievable for the mean squared error have been obtained assuming transmission over the BEC. Moreover, a lower bound on the performance achievable by low-complexity coding schemes has been derived. Such a bound shows that for the mean squared error to decrease exponentially fast in the number of channel uses, the subsequent channel inputs have to depend on a linearly growing number of information bits, i.e., bits of the dyadic expansion of the source vector. Finally, simulation results for linear-complexity coding/decoding schemes have been proposed.

Using finer information-theoretic arguments, most of the results of this paper can be extended to more general discrete memoryless channels. In particular, Theorem 3.1 can be extended to general discrete memoryless channels, providing an upper bound  $\bar{\beta}$  on the achievable error rate which can be written as a function of the sphere-packing exponent of the channel [17, p. 158]. Such a bound turns out to be strictly smaller than the Shannon capacity of the channel, whenever the sphere-packing exponent is finite at rates below capacity; see [7] for similar arguments in the more general context of distributed computation over networks. Theorem 5.2 can be extended to general discrete memoryless channels, showing that exponential error decays require that the  $t$ th channel input depend at least on a linear number of information bits. Using arguments found in [44], Theorem 6.3 for random linear convolutional codes can be extended to the class of discrete memoryless channels which are symmetric with respect to the action of the additive group of some finite field, showing the achievability of the exponential error rate  $\min\{\frac{1}{d}R, \frac{1}{2}E_x(R)\}$ , where  $E_x(R)$  is the expurgated exponent of the channel [17]. It is possible to extend Theorem 6.1 to arbitrary discrete memoryless channels using a random coset approach possibly followed by a quantization as in [17, pp. 206–209], showing that the error rate  $\underline{\beta}'(d, \varepsilon, R) := \min\{\frac{1}{d}R, \frac{1}{2}E_r(R)\}$  is achievable, where  $E_r(R)$  is the random coding exponent of the channel [17]. On arbitrary discrete memoryless channels, linear (or coset) convolutional codes maintain linear encoding complexity, but their maximum a posteriori decoding is known to be an NP-hard problem [1]. The error rate  $\underline{\beta}'(d, \varepsilon, R) := \min\{\frac{1}{d}R, \frac{1}{2}E_r(R)\}$  can be shown to be achievable, on general discrete memoryless channels, by using random nonlinear convolutional codes as in [16, 36, 37]. However, observe that nonlinear convolutional codes require exponential memory for the encoder, while their maximum a posteriori decoding is also an NP-hard problem. Moreover, to our knowledge, no result analogous to Theorem 6.3 is known to hold for nonlinear random convolutional codes.

Some of the questions raised in this paper have been left open. A particularly relevant issue is the analysis and design of linear-complexity coding schemes achieving exponential error rates. Another open problem consists in tightening the upper bound on the achievable error exponent proved in Theorem 3.1 by better exploiting the absence of feedback. Finally, recall that the complexity analysis presented in section 5.2 has been performed while assuming that the joint source-channel encoder could be written as the composition of a fixed dyadic expansion map, and of a channel encoder, and that the a priori distribution of  $x$  is absolutely continuous with respect to Lebesgue's measure. Clearly, the former assumption does not put any restriction on the encoder structure since every joint source-channel encoder can be decomposed in this form, provided that one allows each channel input to depend on a possibly infinite number of bits of the dyadic expansion. Our result on the complexity vs. performance trade-off was obtained by considering the loss in performance induced by restricting

each channel input to depend on a certain finite number of dyadic expansion bits. It would be interesting to investigate how our results can be extended to more general computational models and complexity measures for the encoder and the decoder.

**Appendix A. Proof of Theorem 6.3.** We shall prove Theorem 6.3 by means of so-called code-expurgation arguments. The Hamming weight of a binary string  $\mathbf{y} \in \mathbb{Z}_2^t$  will be denoted by  $w_H(\mathbf{y}) := |\{1 \leq j \leq t : y_j = 1\}|$ . For  $0 \leq j \leq m_t$  and  $h \geq 0$ , let us consider the random variable  $\Upsilon_j^t(h) := |\{\mathbf{y} \in K_j \setminus K_{j+1} : w_H(\tilde{\mathcal{E}}_t \mathbf{y}) = h\}|$ , counting the number of binary strings  $\mathbf{y}$  whose first nonzero bit is the  $(j + 1)$ th and such that  $\tilde{\mathcal{E}}_t \mathbf{y}$  has weight  $h$ . Observe that the causality of  $\phi$  implies that if  $\mathbf{y} \in K_j$ , then  $\tilde{\mathcal{E}}_t \mathbf{y}$  belongs to  $L_j := \text{span}(\delta_s | \lceil (j + 1)/R \rceil \leq s \leq t) \subseteq \mathbb{Z}_2^t$ . Further, since  $\phi \delta_{j+1}$  is uniformly distributed over  $L_j$ , and since the columns of  $\phi$  are independent, we have that if  $\mathbf{y} \in K_j \setminus K_{j+1}$ , then  $\tilde{\mathcal{E}}_t \mathbf{y}$  is uniformly distributed over  $L_j$ . It follows that

$$\mathbb{E}[\Upsilon_j^t(h)] = \sum_{\mathbf{y} \in K_j \setminus K_{j+1}} \mathbb{P}(w_H(\tilde{\mathcal{E}}_t \mathbf{y}) = h) = |K_j \setminus K_{j+1}| \binom{l_j}{h} |L_j|^{-1} \leq 2^{l_j(H(\eta)-1+R)},$$

where  $l_j := (t - \lceil (j + 1)/R \rceil + 1)$  and  $\eta := h/l$ .

For every  $\lambda, \varphi > 0$ , by using the union bound and Markov’s inequality, we can estimate the probability of the event  $F_t := \bigcup_{j=1}^{\lfloor (1-\lambda)Rt \rfloor} \bigcup_{h=0}^{l_j \gamma(R+\varphi)} \{\Upsilon_j^t(h) \geq 1\}$  by  $\mathbb{P}(F_t) \leq \sum_{j,h} \mathbb{E}[\Upsilon_j^t(h)] \leq t^2 2^{-t\lambda\varphi}$ . Then the series  $\sum_n \mathbb{P}(F_n)$  is convergent, and the Borel–Cantelli lemma implies that, with probability one,  $F_n$  occurs finitely many times; i.e., there exists  $t_0 \in \mathbb{N}$  such that  $\Upsilon_j^t(h) = 0$  for all  $h < l_j \gamma(R + \varphi)$ , for all  $t \geq t_0$  and  $1 \leq j \leq \lfloor (1 - \lambda)Rt \rfloor$ . An analogous argument shows that with probability one  $\Upsilon_j^t(h) \leq 2^{l_j(H(\eta)-1+R+\varphi)}$  for all  $l_j \gamma(R + \varphi) \leq h \leq l_j$  for sufficiently large  $t$ .

We are now ready to prove Theorem 6.3. For this, fix  $\lambda, \varphi \in (0, 1)$ , and consider the event  $H_t := \bigcup_{j=1}^{\lfloor (1-\lambda)Rt \rfloor} G_t^j$ , where

$$G_t^j := \bigcup_{h=0}^{l_j \gamma(R+\varphi)} \{\Upsilon_j^t(h) \geq 1\} \bigcup_{h=l_j \gamma(R+\varphi)}^{l_j} \left\{ \Upsilon_j^t(h) \leq 2^{(t-\lceil j/R \rceil)(H(\eta)-1+R+\eta)} \right\}.$$

Then, for  $j = 1, \dots, \lfloor (1 - \lambda)Rt \rfloor$ , the union bound yields the estimation

$$\mathbb{P}(B_j|H_t) \leq \sum_{h=0}^{l_j} \varepsilon^h \mathbb{E}[\Upsilon_j^t(h)|H_t] \leq \sum_{h=l_j \gamma(R+\varphi)}^{l_j} \varepsilon^h 2^{l_j(H(\eta)-1+R+\eta)}.$$

Hence, (6.3) and (6.4) imply that, for  $\kappa := 1 - \lambda)Rt$ ,

$$\begin{aligned} \mathbb{E}[|x - \hat{x}_t|^2 | H_t] &= \sum_{j=1}^{\lfloor \kappa \rfloor} \mathbb{E}[|x - \hat{x}_t|^2 | H_t \cap B_j] \mathbb{P}(B_j | H_t) + \mathbb{E}[|x - \hat{x}_t|^2 \mathbb{1}_{A_{\lfloor \kappa \rfloor}} | H_t] \\ &\leq \sum_{j=1}^{\lfloor \kappa \rfloor} 16d 2^{-2j/d} \sum_{h=l_j \gamma(R+\varphi)}^{l_j} \varepsilon^h 2^{l_j(H(\eta)-1+R+\varphi)} + 16d 2^{-2\lfloor \kappa \rfloor/d} \\ &\leq K' t^2 2^{-t(2\underline{\beta}''(d,\varepsilon,R)-\varphi)} + K'' t^2 2^{-2\kappa} \end{aligned}$$

for some constants  $K', K'' > 0$ . Since, with probability one, there exists  $t_0 \in \mathbb{N}$  such that  $H_t$  occurs for all  $t \geq t_0$ , for all such  $t$  we have

$$\mathbb{E}[|x - \hat{x}_t|^2 | \phi] \leq K' t^2 2^{-t(2\underline{\beta}''(d,\varepsilon,R)-\varphi)} + K'' t^2 2^{-2(1-\lambda)Rt/d}.$$

Finally, the claim follows from the arbitrariness of  $\varphi, \lambda > 0$ . □

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