RESISTANCE-BASED PERFORMANCE ANALYSIS OF THE CONSENSUS ALGORITHM OVER GEOMETRIC GRAPHS*

ENRICO LOVISARI†, FEDERICA GARIN‡, AND SANDRO ZAMPIERI§

Abstract. The performance of the linear consensus algorithm is studied by using a Linear Quadratic (LQ) cost. The objective is to understand how the communication topology influences this algorithm. This is achieved by exploiting an analogy between Markov Chains and electrical resistive networks. Indeed, this permits to uncover the relation between the LQ performance cost and the average effective resistance of a suitable electrical network and, moreover, to show that, if the communication graph fulfils some local properties, then its behavior can be approximated by that of a grid, which is a graph whose associated LQ cost is well-known.

Key words. Multi-agent systems, consensus algorithm, distributed averaging, large-scale graphs

AMS subject classifications. 68R10, 90B10, 94C15, 90B18, 05C50

1. Introduction. The last two decades have witnessed a great effort spent by several scientific communities in the development and in the analysis of multi-agent systems. The large number of simple intercommunicating and interacting entities can be profitably used in order to model a number of rather different applications: just to recall some of them, load balancing [1], coordinated control [2], distributed estimation [3] and distributed calibration for sensor networks [4].

A tool which has been widely proposed for solving these problems is the linear consensus algorithm, which is a powerful and flexible method for obtaining averages in a distributed way with a limited communication effort. This algorithm is used when there is a set of agents, each with a scalar value, and the goal is to drive all agents to reach a common state, under the constraint that agents can only exchange information locally.

To make the concepts more precise, assume that the agents are labeled by the elements of the set $V = \{1, \dots, N\}$ and that the graph $\mathcal{G} = (V, \mathcal{E})$, where $\mathcal{E} \subseteq V \times V$, describes which communication links between the agents are allowed. More precisely, we say that $(u, v) \in \mathcal{E}$ if and only if the agent v can send information to u. The graph $\mathcal{G}$ is called the communication graph. In the linear consensus algorithm, at each iteration, the agents send their current state to their neighbors, and then update their state as a suitable convex combination of the received messages. More precisely, if $x_u(t)$ denotes the state of the agent $u \in V$ at time $t \in \mathbb{N}$, then

$$x_u(t+1) = \sum_{v \in V} P_{uv} x_v(t)\,, \qquad (1.1)$$

where P_{uv} are the entries of a stochastic[1] matrix P. In a more compact form we can

*The research leading to these results has received funding from the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreements n. 257462 HYCON2 Network of excellence and n. 223866 FeedNetBack.

†DEI (Department of Information Engineering), Università di Padova, Via Gradenigo 6/b, 35131 Padova, Italy, `zampi@dei.unipd.it`

‡INRIA Grenoble Rhône-Alpes, 655 av. de l'Europe, Montbonnot, 38334 Saint-Ismier cedex, France, `federica.garin@inria.fr`

§DEI (Department of Information Engineering), Università di Padova, Via Gradenigo 6/b, 35131 Padova, Italy, `zampi@dei.unipd.it`

[1] A matrix P is said to be stochastic if $P_{uv} \geq 0$ for all $u, v \in V$ and $\sum_{v \in V} P_{uv} = 1$ for all $u \in V$.

write

$$\boldsymbol{x}(t+1) = P\boldsymbol{x}(t) \tag{1.2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^N$ denotes the vector collecting all agents' states. The constraint imposed by the communication graph $\mathcal{G}$ is enforced by requiring that $\mathcal{G}_P$ is a subgraph of $\mathcal{G}$, where $\mathcal{G}_P = (V, \mathcal{E}_P)$ is the *graph associated with the matrix* P, defined assuming that $(u, v) \in \mathcal{E}_P$ if and only if $P_{uv} \neq 0$.

The stochastic matrix P is said to be *irreducible* if the associated graph is strongly connected, namely, for all $u, v \in V$, there exists a path in $\mathcal{G}_P$ connecting u to v, and it is said to be *aperiodic* if the greatest common divisor of the lengths of all cycles in $\mathcal{G}_P$ is one. Notice that the presence of a self-loop, namely a $P_{uu} > 0$ for some $u \in V$, ensures aperiodicity. As it is well known from Frobenius-Perron theory [5], if P is irreducible and aperiodic, then P has the eigenvalue 1 with algebraic multiplicity 1, and all other eigenvalues have absolute value strictly smaller than 1 and so we have that

$$P^t \overset{t\to\infty}{\longrightarrow} \mathbf{1}\boldsymbol{\pi}^T$$

where the vector $\boldsymbol{\pi}$ is the *invariant measure* of the matrix P, namely the left eigenvector of P corresponding to the eigenvalue 1, properly scaled so as to have $\sum_u \pi_u = 1$. Consequently, under these hypotheses, the states of the consensus algorithm (1.2) converge to the same value $x_u(t) \overset{t\to\infty}{\longrightarrow} \alpha$, where $\alpha = \boldsymbol{\pi}^T\boldsymbol{x}(0)$.

The convergence to the consensus value is exponential with exponent equal to the second largest eigenvalue of the matrix P

$$\rho(P) = \sup\{|\lambda| : \lambda \in \sigma(P), \lambda \neq 1\}$$

where $\sigma(P)$ is the spectrum of P. For this reason, the value $\rho(P)$ is a classical performance cost for the algorithm. Indeed, the closer is $\rho(P)$ to zero, the faster is the algorithm. However, as recent papers have pointed out [6, 7], this performance index on P is not the only possible choice for evaluating the performance of the algorithm. Different costs arise from different specific problems where the consensus algorithm is used. Moreover, it can be shown [8, 9] that, by considering different performance indices, it is possible to obtain different optimal graph topologies.

In this paper, we propose a Linear Quadratic (LQ) cost which is a performance index widely used in the control community. To evaluate how fast P^t converges to its limit value $\mathbf{1}\boldsymbol{\pi}^T$ we propose the index

$$J(P) := \frac{1}{N}\sum_{t\geq 0} ||P^t - \mathbf{1}\boldsymbol{\pi}^T||_F^2 = \frac{1}{N}\,\mathrm{trace}\left[\sum_{t\geq 0}(I - \boldsymbol{\pi}\mathbf{1}^T)(P^T)^t P^t (I - \mathbf{1}\boldsymbol{\pi}^T)\right].$$

where $\|M\|_{\mathrm{F}} := \sqrt{\mathrm{trace}(MM^T)}$ is the Frobenius norm of a matrix and where we added the normalizing factor $1/N$ for reasons which will be clarified in the following.

This cost appears in two different contexts. Assume first that we want to evaluate the speed of convergence of the consensus algorithm by the ℓ^2 norm of the transient, namely

$$\frac{1}{N}\sum_{t\geq 0}\left[\|\boldsymbol{x}(t) - \boldsymbol{x}(\infty)\|^2\right].$$

Notice that this ℓ^2 norm will depend on the initial condition $\boldsymbol{x}(0)$. For this reason, we assume that the initial condition is a random variable with zero-mean and covariance matrix $\mathbb{E}\left[\boldsymbol{x}(0)\boldsymbol{x}(0)^T\right] = I$. We can now consider the expected value of the ℓ^2 norm of the transient which is now a function only of the matrix P. Indeed, by some simple computations [8] it can be shown that

$$\mathbb{E}\left[\frac{1}{N}\sum_{t\geq 0}\|\boldsymbol{x}(t)-\boldsymbol{x}(\infty)\|^2\right] = J(P)\,.$$

The cost $J(P)$ appears also in the context of noisy consensus [6, 8, 10]. Consider a network of agents implementing the consensus algorithm, in which update is affected by additive noise, so that the actual update of the state is the following

$$\boldsymbol{x}(t+1) = P\boldsymbol{x}(t) + \boldsymbol{n}(t)\,,$$

where $\boldsymbol{n}(t)$ is a random white process. Assume that $\mathbb{E}[\boldsymbol{n}(t)] = 0$ and $\mathbb{E}[\boldsymbol{n}(t)\boldsymbol{n}(t)^T] = I$ for all $t \in \mathbb{N}$. Assume that the initial condition is random and that it is uncorrelated from the noise process. We are interested in the dispersion of $\boldsymbol{x}(t)$. If we measure it by evaluating the displacement of $\boldsymbol{x}(t)$ from the weighted average $\sum_i \pi_i x_i(t)$, namely by introducing the vector

$$\boldsymbol{e}(t) = (I - \mathbf{1}\boldsymbol{\pi}^T)\boldsymbol{x}(t),$$

then it can be shown that

$$\lim_{t\to\infty}\mathbb{E}\left[\|\boldsymbol{e}(t)\|^2\right] = J(P),$$

Thus, the proposed LQ cost also characterizes the spreading of the asymptotic value of the state vector around its weighted average in a noisy network.

It is possible to consider the problem of determining the matrix P satisfying a constraint and minimizing the index $J(P)$. In this paper we will instead consider a different problem. Indeed we will try to provide estimates of $J(P)$ which permit to understand how this index depends on the structure of P and more precisely on the topological properties of the graph $\mathcal{G}_P$. More precisely we will be able to unveil this dependence, proving that $J(P)$ is related to the effective resistance of a suitable electrical network. This geometric parameter depends on the topology only, and not on the particular entries of P. Since the electrical analogy holds only if P is reversible [11], in this paper we will restrict to this class of matrices. An important subclass of reversible matrices is the one of symmetric matrices, for which stronger and simpler results hold true. The results for the symmetric case were already described, without the proof details, in our paper [12].

Using these results we will show that, analogously to what happens for convergence rate [13, 14], under some assumptions, a large class of graphs, called geometric and which can be seen as perturbed grids, exhibit a particular behavior of the cost $J(P)$ as a function of the number of nodes which depends on the geometric dimension of the graph. In particular, if the graph has geometric dimension one, namely it is a geometric graph on a segment, then $J(P)$ grows linearly in the number of nodes, while, if the graph has geometric dimension two, namely it is a geometric graph on a square, then $J(P)$ grows logarithmically in the number of nodes. Finally, if the graph has geometric dimension three (or more), namely it is a geometric graph on a cube, then $J(P)$ is bounded from above by a constant independent on the number of nodes. This result is based on (and extends) an analogous result [6] which holds for torus graphs. In this way we show that the spatial invariance of torus graphs is not a necessary requirement for having this kind of behavior of $J(P)$.

The paper is organized as follows. In Section 2 we give some basic notions on reversible consensus matrices. Section 3 is devoted to recalling the analogy among reversible consensus matrices and electrical networks, which allows us to state the main results. In Section 4 we present a particularly appealing application of our results, showing that the performance cost in a family of geometric graphs only depends on the dimensionality of the graphs. The proofs of the results are postponed to Section 5 and Section 6. Finally, in Section 7 we draw conclusions.

1.1. Notation. In this paper we will denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}_+$ the set of nonnegative real numbers. Vectors will be denoted by boldface letters, e.g. $\boldsymbol{x}$, while entries of vectors and scalars will be in italic font. We denote by $\mathbf{1}$ the column vector with all entries equal to 1 and of suitable dimension, and by $\boldsymbol{e}_u$ the u-th element of the canonical basis of $\mathbb{R}^N$, i.e., a vector whose u-th entry is 1 and all other entries are 0. Given $\boldsymbol{v} \in \mathbb{R}^N$, the symbol $\operatorname{diag} \boldsymbol{v}$ where denotes a $N \times N$ diagonal matrix whose (k, k)-th entry is the k-th entry of $\boldsymbol{v}$. Given $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^N$ and a positive definite matrix M, we will denote the inner product and the norm weighted by M as $\langle \boldsymbol{v}, \boldsymbol{w} \rangle_M := \boldsymbol{v}^T M \boldsymbol{w}$ and $\|\boldsymbol{v}\|_M := \sqrt{\boldsymbol{v}^T M \boldsymbol{v}}$, respectively.

2. Preliminaries. In this paper we will analyze the cost function $J(P)$ when P is an irreducible and aperiodic stochastic matrix, so that $J(P)$ is finite. In fact, aperiodicity will be the consequence of the stronger assumption which imposes that the diagonal elements of P are all positive. Observe that this condition is not restrictive for the consensus algorithm as it assumes only that in the state-update (1.1) each agent gives to its own current state a positive weight and this does not requires additional communication. For this reason, throughout the paper we will use the following definition.

DEFINITION 2.0.1. *We say that a matrix P is a* consensus matrix *if it stochastic and irreducible, and it satisfies $P_{uu} > 0$ for all u.*

Recall [5] that a consensus matrix has a dominant eigenvalue 1 with algebraic multiplicity 1. As already pointed out, the corresponding left eigenvector $\boldsymbol{\pi}$, normalized so that $\sum_u \pi_u = 1$, is called the invariant measure of P (a name coming from the interpretation of P as the transition probability matrix of a Markov chain) and it has all entries which are strictly positive.

An useful object is the Laplacian of a matrix P, which is defined as $L := I - P \in \mathbb{R}^{N \times N}$. It is immediate to check that $L\mathbf{1} = \mathbf{0}$ and $L_{uv} \leq 0, \forall\, u \neq v$. Moreover, since P has 1 as eigenvalue with algebraic multiplicity one, then $\dim(\ker L) = 1$. One of the most important classes of consensus matrices is that of *reversible* matrices, a name which comes, again, from the definition of reversible Markov chains [15].

DEFINITION 2.0.2. *Let P be a consensus matrix with invariant measure $\boldsymbol{\pi}$. Then P is said to be reversible if*

$$\pi_u P_{uv} = \pi_v P_{vu}, \ \forall\, (u, v) \in V \times V$$

or equivalently if

$$\Pi P = P^T \Pi, \tag{2.1}$$

where $\Pi = \operatorname{diag} \boldsymbol{\pi}$.

REMARK 2.0.1. *The graph associated with a consensus matrix P in general is a directed graph. However, the assumption that the consensus matrix is reversible implies that the graph if undirected, i.e., $(u, v) \in \mathcal{E} \iff (v, u) \in \mathcal{E}$. The proof comes immediately from Eq. (2.1) and the property that $\pi_u > 0$ for all u, which imply that $P_{uv} > 0$ if and only if $P_{vu} > 0$.*

For this reason, in the following we will always assume that the communication graph $\mathcal{G}$ is undirected. Also recall that in our definition a consensus matrix has non-zero diagonal elements, and thus we will consider only graphs having a self-loop at each node. However, consistently with previous literature of consensus, we will define the neighborhood of a node $u \in \mathcal{G}$ to be the set of its neighbors, except u itself

$$\mathcal{N}_u := \{v \in V : v \neq u, (u, v) \in \mathcal{E}\}$$

and we will define the degree of u to be the cardinality $|\mathcal{N}_u|$ of its neighborhood.

Notice that Eq. (2.1) states that reversible matrices are self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_\Pi$, or, equivalently, that the matrix $\Pi^{1/2} P \Pi^{-1/2}$ is symmetric. This implies that P has real eigenvalues and that it has N independent eigenvectors.

Now we briefly recall the notion of Green matrix of a consensus matrix P, which is also known as fundamental matrix in the Markov chains literature. Here we concentrate only on the results that are needed in the paper. A more complete list of the properties of the fundamental matrix can be found in [16].

DEFINITION 2.0.3. *Let P be a consensus matrix, with invariant measure $\boldsymbol{\pi}$. The Green matrix G of P is defined as*

$$G := \sum_{t \geq 0} (P^t - \mathbf{1}\boldsymbol{\pi}^T)\,. \tag{2.2}$$

The Green matrix plays a fundamental role in this paper due to its property of being almost an inverse of the Laplacian, in these sense that

$$G + \mathbf{1}\boldsymbol{\pi}^T = (L + \mathbf{1}\boldsymbol{\pi}^T)^{-1}.$$

The above expression is easy to verify, and implies the following equation, which will be useful later on

$$\begin{bmatrix} G & \mathbf{1} \end{bmatrix} \begin{bmatrix} L \\ \boldsymbol{\pi}^T \end{bmatrix} = I\,. \tag{2.3}$$

3. Reversible consensus matrices and electrical networks. In this section we present electrical networks, their relation with consensus matrices, and the well-known notion of effective resistance. Making use of these notions, we state our main results, Theorem 3.2.1 and Theorem 3.2.2, which give useful bounds of the cost $J(P)$ of a reversible consensus matrix P.

3.1. The electrical analogy. The analogy between consensus matrices, or Markov chains, and resistive electrical networks dates back to the work of Doyle and Snell [11]. It is a powerful tool which gives strong intuitions on the behavior of the chain on the basis of the physics of electrical networks, as well as permitting simple and clear proofs for many results. Our interest is mainly related to the possibility to rewrite the LQ cost we are interested in in terms of a geometric parameter, the average effective resistance. To this respect we are strongly indebted in terms of inspiration to the papers by Barooah and Hespanha [17, 18, 19], from which we took many results we state here without a proof. Effective resistances also arise as a performance metric for clock synchronization algorithms in [20, 21], and methods for its minimization are proposed in [22]. To conclude, in [23] the effective resistance is computed in terms of the eigenvalues of the Laplacian matrix in the symmetric case.

3.1.1. Electrical networks. A resistive electrical network is a graph in which pairs of nodes are connected by resistors. A resistive electrical network is therefore determined by a symmetric matrix C with non-negative entries which tells for each pair of nodes u, v which is the conductance of the resistor connecting those two nodes. A resistive electrical network is said to be connected if the graph $\mathcal{G}_C$ associated with C is connected.

In order to describe the current flowing in the electrical network and to write Kirchoff's and Ohm's laws, we choose (arbitrarily) a conventional orientation for each edge of the undirected graph $\mathcal{G}$, so that current will be denoted as positive when flowing consistently with the direction of the edge and negative otherwise. To this aim, for any pair of nodes u, v, such that $u \neq v$ and $C_{uv} = C_{vu} \neq 0$, we choose either (u, v) or (v, u) in $V \times V$. Let $\mathcal{E} \subseteq V \times V$ be the set of directed edges formed in this way and let M be the number of edges. Define the incidence matrix $B \in \mathbb{R}^{M \times N}$ as follows: order the edges from 1 to M and let for any $e \in \{1, \dots, M\}$ and $u \in \{1, \dots, N\}$

$$B_{eu} = \begin{cases} -1 & \text{if the edge } e \text{ is } (v,\, u) \text{ for some } v \neq u, \\ 1 & \text{if the edge } e \text{ is } (u,\, v) \text{ for some } v \neq u, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

We define the diagonal matrix $\mathcal{C} \in \mathbb{R}^{M \times M}$ having the conductance of the edge e as the entry in position (e, e). The relation between $\mathcal{C}$ and C is easily obtained as

$$B^T \mathcal{C} B = \operatorname{diag} C\mathbf{1} - C\,, \tag{3.2}$$

namely

$$[B^T \mathcal{C} B]_{uv} = \begin{cases} C_u & \text{if } u = v, \\ -C_{uv} & \text{if } (u,\, v) \in \mathcal{E}, \\ 0 & \text{if } (u,\, v) \notin \mathcal{E}, \end{cases}$$

where $C_u := \sum_{v \in V} C_{uv}$.

Let $\boldsymbol{i} \in \mathbb{R}^N$ be given such that $\boldsymbol{i}^T \mathbf{1} = 0$, and interpret the k-th entry of $\boldsymbol{i}$ as the current which is injected (or extracted if negative in sign) into the k-th node of the network from an external source. We denote by $\boldsymbol{j} \in \mathbb{R}^M$ and $\boldsymbol{v} \in \mathbb{R}^N$, respectively, the current flows on the edges and the potentials at the nodes which are produced in the network by injecting the current $\boldsymbol{i}$, with the convention that j_e, $e \in \mathcal{E}$, is positive when the current flows in the direction of e. The previously defined matrices B and $\mathcal{C}$ allow us to compactly write Kirchhoff's node law and Ohm's law as the system as follows

$$\begin{cases} B^T \boldsymbol{j} = \boldsymbol{i}\,, \\ \mathcal{C} B \boldsymbol{v} = \boldsymbol{j}\,, \end{cases} \tag{3.3}$$

where the first equation states that the total current flow entering into each node equals the total flow exiting from it (Kirchhoff's current law), while the second equation represents Ohm's law, $C_{uu'}(v_u - v_{u'}) = j_e$ for all $e = (u,\, u') \in \mathcal{E}$.

Solving the electrical network means finding the solutions $\boldsymbol{j}$ and $\boldsymbol{v}$ of Eq. (3.3), in particular finding the solution $\boldsymbol{v}$ of the following electrical equation

$$B^T\mathcal{C}B\boldsymbol{v} = \boldsymbol{i}\,. \tag{3.4}$$

It is well known (see, e.g., [11]) that a solution exists and, for a connected network, is unique up to a constant additive term for $\boldsymbol{v}$, i.e., the differences $v_u - v_{u'}$ are uniquely defined. In the next subsection, we will give an explicit expression for the solutions $\boldsymbol{v}$, involving the Green matrix of the associated reversible consensus matrix.

Given a connected electrical network with conductance matrix C, the *effective resistance* between two nodes u, u' is defined to be

$$\mathcal{R}_{uu'}(C) := v_u - v_{u'}$$

where we impose $\boldsymbol{i} = \boldsymbol{e}_u - \boldsymbol{e}_{u'}$ and $\boldsymbol{v}$ is any solution of the corresponding electrical equation (3.4)namely, $\boldsymbol{v}$ is the potential at each node of the network in the case when a unit current is injected at node u and extracted ad node u'. Finally the *average effective resistance* of the electrical network is defined as

$$\bar{\mathcal{R}}(C) := \frac{1}{2N^2}\sum_{u,\,u'\in V}\mathcal{R}_{uu'}(C). \tag{3.5}$$

Given a connected undirected graph $\mathcal{G}$, in the following we will use the notations $\mathcal{R}_{uu'}(\mathcal{G})$ and $\bar{\mathcal{R}}(\mathcal{G})$, as the effective resistance and the average effective resistance associated with the electrical network having conductance equal to 1 for all the edges of $\mathcal{G}$ and conductance equal to 0 otherwise.

3.1.2. Electrical network associated with a consensus matrix. There is a way to obtain a one to one relation between reversible consensus matrices and connected resistive electrical networks with some fixed total conductance (i.e., sum of the conductances of all edges). Let P be a reversible consensus matrix and let

$$\Phi(P) := N\Pi P$$

where $\Pi = \mathrm{diag}(\boldsymbol{\pi})$ and $\boldsymbol{\pi}$ is the invariant measure of P. It is clear that $\Phi(P)$ is the conductance matrix of a connected resistive network. It can be shown that the map Φ is injective. Indeed, if P_1, P_2 are reversible consensus matrices and if $\Phi(P_1) = \Phi(P_2)$, then $\mathrm{diag}(\boldsymbol{\pi}_1)P_1 = \mathrm{diag}(\boldsymbol{\pi}_2)P_2$. Multiplying on the right both members by $\mathbf{1}$ we obtain that $\boldsymbol{\pi}_1 = \boldsymbol{\pi}_2$ and consequently $P_1 = P_2$. We show now that $\mathrm{Im}(\Phi) = \mathcal{S}$, where

$$\mathcal{S} := \{C \in \mathbb{R}_+^{N\times N} : C = C^T,\, C_{uu} > 0\ \forall\ u \in V,\, \mathcal{G}_C \text{ is connected, and } \mathbf{1}^T C\mathbf{1} = N\}$$

Clearly $\mathrm{Im}(\Phi) \subseteq \mathcal{S}$. To prove the equality, for any given $C \in \mathcal{S}$ we consider

$$P = (\mathrm{diag}(C\mathbf{1}))^{-1}\, C \tag{3.6}$$

and we show that P is a reversible consensus matrix such that $\Phi(P) = C$. It is straightforward to see that $P\mathbf{1} = \mathbf{1}$, $P_{uu} > 0$, $\forall\ u \in V$, and that $\mathcal{G}_P$ is connected. To prove that P is reversible and that $\Phi(P) = P$, the key remark is that

$$\boldsymbol{\pi} = \frac{1}{N}C\mathbf{1}$$

is the invariant measure of P, and thus $\Pi = \frac{1}{N}\,\mathrm{diag}(C\mathbf{1})$. This immediately implies that $\Phi(P) = C$. The reversibility of P is then proved by using the symmetry of C: $\Pi P = \frac{1}{N}C = \frac{1}{N}C^T = P^T\Pi$. In this way, we have proved not only that Φ is bijective on $\mathcal{S}$, but also that (3.6) provides the inverse of Φ over $\mathcal{S}$.

Consider now a reversible consensus matrix P and its associated conductance matrix $C := \Phi(P)$. Let moreover B and $\mathcal{C}$ be the matrices associated with the resistive electrical network with conductance C, as defined above. Notice that the Laplacian matrix $L := I - P = \frac{1}{N}\Pi^{-1}B^T\mathcal{C}B$ and so the electrical equation 3.4 is equivalent to

$$L\boldsymbol{v} = \tfrac{1}{N}\Pi^{-1}\boldsymbol{i}. \tag{3.7}$$

The network being connected, $\ker L = \{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}$, and thus, for any $\boldsymbol{i}$ such that $\boldsymbol{i}^T\mathbf{1} = 0$, Eq. (3.7) has infinitely many solutions, of the form $\boldsymbol{v} + \alpha\mathbf{1}$ for some real constant α, where $\boldsymbol{v}$ is a particular solution. In our setting it is convenient to find $\boldsymbol{v}$ which satisfies the following constraint

$$\boldsymbol{\pi}^T\boldsymbol{v} = 0,$$

which means that we need to solve the equation

$$\begin{bmatrix} L \\ \boldsymbol{\pi}^T \end{bmatrix}\boldsymbol{v} = \begin{bmatrix} \frac{1}{N}\Pi^{-1}\boldsymbol{i} \\ 0 \end{bmatrix}.$$

Thanks to Eq. (2.3), the solution can be explicitly written by using the Green matrix G associated with P, as follows

$$\boldsymbol{v} = \begin{bmatrix} G & \mathbf{1} \end{bmatrix}\begin{bmatrix} \frac{1}{N}\Pi^{-1}\boldsymbol{i} \\ 0 \end{bmatrix} = \tfrac{1}{N}G\Pi^{-1}\boldsymbol{i}\,.$$

Consequently, we can obtain the effective resistance as follows

$$\mathcal{R}_{uu'}(C) = \tfrac{1}{N}(\boldsymbol{e}_u - \boldsymbol{e}_{u'})^T G\Pi^{-1}(\boldsymbol{e}_u - \boldsymbol{e}_{u'})\,. \tag{3.8}$$

3.2. LQ cost and effective resistance. This section is devoted to our main results on the relation between the LQ cost $J(P)$ for a reversible consensus matrix P and the average effective resistance of a suitable electrical network. The results are then formulated in the special case of symmetric consensus matrices, since for this case they turns out to be clearer and more readable. The proofs are given in Section 5.

Consider a reversible consensus matrix P and let $\boldsymbol{\pi}$ be its invariant measure. Build the electrical network associated with the matrix P^2 as suggested in Sec. 3.1.2, namely build the matrix of conductances

$$C := \Phi(P^2) = N\Pi P^2. \tag{3.9}$$

In the particular case in which P is symmetric we have that $C = P^2$. The following theorem allows us to estimate the cost in terms of the average effective resistance of this electrical network, and of quantities depending on the elements of the invariant measure of P.

THEOREM 3.2.1. *Let $P \in \mathbb{R}^{N\times N}$ be a reversible consensus matrix and $\boldsymbol{\pi}$ its invariant measure, and C the matrix of conductances defined in Eq. (3.9). Then it holds*

$$\frac{\pi_{\min}^3 N^2}{\pi_{\max}}\bar{\mathcal{R}}(C) \leq J(P) \leq \frac{\pi_{\max}^3 N^2}{\pi_{\min}}\bar{\mathcal{R}}(C)\,,$$

where $\pi_{\min}$ *and* $\pi_{\max}$ *are respectively the minimum and maximum entries of* $\boldsymbol{\pi}$*.*

In the particular case of symmetric matrix we have the following corollary which is a straightforward consequence of the previous theorem.

COROLLARY 3.1. *Let* $P \in \mathbb{R}^{N\times N}$ *be a symmetric consensus matrix, and* C *the matrix of conductances defined in Eq. (3.9). Then it holds*

$$J(P) = \bar{\mathcal{R}}(C).$$

The previous results catch the dependance of the cost on the electrical network built from P^2. The following theorem allows us to write the cost $J(P)$ in terms of the effective resistance of the graph associated with P only, regardless of the particular entries of the matrix.

THEOREM 3.2.2. *Let* P *be a reversible consensus matrix with invariant measure* $\boldsymbol{\pi}$ *and let* $\mathcal{G}$ *be the graph associated with* P*. Assume that all the non-zero entries of* P *belong to the interval* $[p_{\min}, p_{\max}]$*, and that the degree of any node is bounded from above by an integer* δ*. Then,*

$$\frac{\pi_{\min}^3 N}{8p_{\max}^2\delta^2\pi_{\max}^2}\bar{\mathcal{R}}(\mathcal{G}) \leq J(P) \leq \frac{\pi_{\max}^3 N}{p_{\min}^2\pi_{\min}^2}\bar{\mathcal{R}}(\mathcal{G})\,.$$

A simpler result can be obtained for symmetric matrices as a straightforward consequence of the previous theorem.

COROLLARY 3.2. *Let* P *be a symmetric consensus matrix associated with a graph* $\mathcal{G}$*. Assume that all the non-zero entries of* P *belong to the interval* $[p_{\min}, p_{\max}]$*, and that the degree of any node is bounded from above by an integer* δ*. Then,*

$$\frac{1}{8p_{\max}^2\delta^2}\bar{\mathcal{R}}(\mathcal{G}) \leq J(P) \leq \frac{1}{p_{\min}^2}\bar{\mathcal{R}}(\mathcal{G})\,.$$

These last two results can be used to estimate the proposed LQ-cost in terms of the effective resistance of graphs only, as we will show in Section 4.

3.3. LQ cost and network dimension. One of the most important problems in the design of a sensor network is to dimension it, namely to decide how many sensors we need to deploy for obtaining a given performance. From this point of view, it is very important to understand how our cost function scales in terms of the number N of nodes in a sequence of graphs of growing size, belonging to a given family. The results in the previous sections can be used to achieve this goal.

Consider in fact a sequence of graphs $\{\mathcal{G}_N\}_N$, and assume that $f(N) = \bar{\mathcal{R}}(\mathcal{G}_N)$ is a known function of N. Assume that the degree of any node of each $\mathcal{G}_N$ is uniformly bounded from above by a positive integer δ. At first, assume to build a sequence of symmetric matrices P_N, each one consistent with the corresponding $\mathcal{G}_N$, and such that if $[P_N]_{ij} \neq 0$, then $p_{\min} \leq [P_N]_{ij} \leq p_{\max}$, for all N. Then we immediately obtain by Corollary 3.2 that the asymptotic scaling of $J(P_N)$ for $N \to \infty$ is given by $f(N)$, up to some multiplicative constant. Notice that the above-mentioned assumptions on the family P_N are satisfied, for example, by one of the most popular consensus protocols on networks with bidirectional communication (i.e., on undirected graphs), defined as follows

$$x_u(t+1) = x_u(t) + \varepsilon \sum_{v\in\mathcal{N}_u}(x_v(t) - x_u(t))$$

for a sufficiently small $\varepsilon > 0$.

If we relax the assumption that all P_N's are symmetric, and we consider a family of reversible matrices P_N, each one consistent with the corresponding $\mathcal{G}_N$, the uniform bound from above and from below on the non-zero entries $[P_N]_{ij}$ is not enough to ensure that the asymptotic behavior of $J(P_N)$ is given by $f(N)$. In fact, once we denote by $\boldsymbol{\pi}_N$ the invariant measure of P_N, and by $\pi_{N,\min}$ and $\pi_{N,\max}$ respectively the minimum and maximum value of the entries of $\boldsymbol{\pi}_N$, we need the further assumption that the sequences $N\pi_{N,\min}$ and $N\pi_{N,\max}$ are uniformly bounded from above and below by constants independent of N. Under this assumption, Theorem 3.2.2 clearly ensures that the asymptotic behavior of $J(P_N)$ is given by $f(N)$. Although the assumption requiring that $N\pi_{N,\min}$ and $N\pi_{N,\max}$ are uniformly bounded can be rather difficult to check for the general reversible consensus matrices, we can easily see that it holds true in the following important example of consensus iteration (known as 'simple random walk' or 'uniform weights')

$$x_u(t+1) = \frac{1}{|N_u|+1}\Big(x_u(t) + \sum_{v\in\mathcal{N}_u} x_v(t)\Big).$$

Indeed, it can be shown that, if the graph is undirected, then the consensus matrix in this case is reversible, and that the invariant measure $\boldsymbol{\pi}$ has components

$$\pi_u = \frac{|N_u|+1}{\sum_{v\in V}(|N_v|+1)}$$

Under the assumption that the degree of any node is bounded by a value δ, it is clear that the entries of the consensus matrix belong to the interval $[\frac{1}{\delta+1}, \frac{1}{2}]$. One can also check that each entry of the invariant measure lies in the interval $[\frac{2}{(\delta+1)N}, \frac{\delta+1}{2N}]$, and hence the assumptions are satisfied.

We conclude this remark noticing that the assumption requiring that $N\pi_{N,\min}$ and $N\pi_{N,\max}$ are uniformly bounded is not implied by the other assumptions that P_N are reversible consensus matrices whose entries belong to a fixed interval $[p_{\min}, p_{\max}]$ and are consistent with graphs with degrees uniformly bounded by δ. This is proved in the example showed in figure 3.1. One can check that the corresponding consensus matrix is reversible and that the invariant measure $\boldsymbol{\pi}$ is such that $\pi_k = \alpha(a/b)^{k-1}$, where α is a suitable normalizing factor. If we assume that $a > b$, then $\pi_{N,\min} = \pi_1 = \alpha$ and $\pi_{N,\max} = \pi_N = \alpha(a/b)^{N-1}$. In this case $N\pi_{N,\min}$ and $N\pi_{N,\max}$ cannot be uniformly bounded from below and above, because if this were the case, then also the ratio $N\pi_{N,\max}/N\pi_{N,\min}$ would be uniformly bounded from below and above, but this is not possible, as $N\pi_{N,\max}/N\pi_{N,\min} = \pi_{N,\max}/\pi_{N,\min} = (a/b)^{N-1}$.

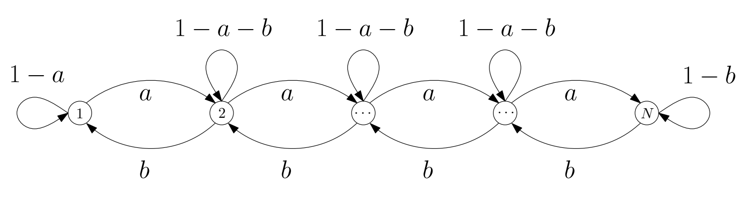

FIGURE 3.1. *Example: a family of growing lines. We assume that* $a > 0$, $b > 0$ *and* $a + b < 1$.

4. Application to geometric graphs. Let $\mathcal{G} = (V, \mathcal{E})$ be a connected undirected graph such that $V \subset Q$ where $Q = [0, \ell]^d \subseteq \mathbb{R}^d$ and $|V| = N$. Namely, the nodes of the graph are deployed in some d-dimensional hypercube of side length equal to ℓ. Given such a graph and two nodes $u,\, v \in V$, we denote by $d_{\mathrm{E}}(u,\, v)$ the Euclidean distance between u and v in $\mathbb{R}^d$, namely

$$d_{\mathrm{E}}(u,\, v) = \sqrt{\sum_{k=1}^{d} (u_k - v_k)^2}.$$

Following [18], we define following parameters associated with $\mathcal{G}$:

- the minimum Euclidean distance between any two nodes
$$s = \min_{u,\, v \in V,\, u \neq v} \{d_{\mathrm{E}}(u,\, v)\}\,; \tag{4.1}$$
- the maximum Euclidean distance between any two connected nodes
$$r = \max_{(u,\, v) \in \mathcal{E}} \{d_{\mathrm{E}}(u,\, v)\}\,; \tag{4.2}$$
- the radius of the largest ball centered in Q not containing any node of the graph
$$\gamma = \max\{r |\, B(x,\, r) \cap V = \emptyset,\, \forall\, x \in Q\}\,, \tag{4.3}$$
where $B(x,\, r)$ denotes the d-dimensional ball centered in $x \in \mathbb{R}^d$ and with radius r;
- the minimum ratio between the Euclidean distance of two nodes and their graphical distance[2]
$$\rho = \min\left\{\frac{d_{\mathrm{E}}(u,\, v)}{d_{\mathcal{G}}(u,\, v)} |\, (u,\, v) \in V \times V\right\}. \tag{4.4}$$

One important example of geometric graph is given by the *regular grid* with dimension d. This is defined as the geometric graph with $N = n^d$ nodes, lying in a hypercube of edge length $\ell = n - 1$, whose nodes have coordinates $(i_1, \ldots, i_d)$, where $i_1, \ldots, i_d \in \{0, 1, \ldots, n-1\}$, and in which two nodes u, v are connected if and only if $d_{\mathrm{E}}(u, v) = 1$.

4.1. Geometric graphs: main result. The following theorem is the main result of this section.

THEOREM 4.1.1. *Let $P \in \mathbb{R}^{N \times N}$ be a reversible consensus matrix with invariant measure $\boldsymbol{\pi}$, associated with a graph $\mathcal{G} = (V,\, \mathcal{E})$. Assume that all the non-zero entries of P belong to the interval $[p_{\min},\, p_{\max}]$ and that $\mathcal{G}$ is a geometric graph with parameters $(s,\, r,\, \gamma,\, \rho)$ and nodes lying in $Q = [0, \ell]^d$ in which $\gamma < \ell/4$. Then*

$$k_1 + q_1 f_d(N) \leq J(P) \leq k_2 + q_2 f_d(N)\,,$$

where

$$f_d(N) = \begin{cases} N & \text{if } d = 1, \\ \log N & \text{if } d = 2, \\ 1 & \text{if } d \geq 3, \end{cases} \tag{4.5}$$

[2] The graphical distance $d_{\mathcal{G}}(u,\, v)$ between u and v is defined as the length (i.e., number of edges) of the minimum path connecting u and v.

and where k_1, k_2, q_1 and q_2 depend on $p_{\max}$, $p_{\min}$, δ, d, on $\pi_{\min}N$ and $\pi_{\max}N$ and on the parameters s, r, γ, ρ of the geometric graph.

One of the most important consequences of this result is the fact that a d-dimensional regular grid has the same behavior of the LQ cost as a function of N of a irregular geometric graph. This implies that the behavior of the LQ cost as a function of N is essentially captured by the dimensionality, rather than by the symmetry, exactly as it happens for the rate of convergence towards consensus [13, 14].

4.2. Growing families of geometric graphs. Theorem 4.1.1 holds for each given geometric graph with parameters s, r, γ and ρ. However, very much similarly to what is done in Section 4.1, we want to use this result in order to capture the asymptotic behavior in term of the dimension of the network.

Consider a growing family of geometric graphs $\mathcal{G}_N$ with $\mathcal{G}_N = (V_N, \mathcal{E}_N)$ and $|V_N| = N$. Let each $\mathcal{G}_N$ be a geometric graph with parameters s_N, r_N, γ_N and ρ_N. Assume that there exist parameters s, r, γ and ρ, which we call *the geometric parameters of the family*, such that

$$s_N \geq s,\, r_N \leq r,\, \gamma_N \leq \gamma,\, \rho_N \geq \rho, \qquad \forall\, N\ . \tag{4.6}$$

Let P_N be the reversible consensus matrix associated with $\mathcal{G}_N$ and with invariant measure $\boldsymbol{\pi}_N$. First of all, we need that all the non-zero entries of P_N belong to the interval $[p_{\min}, p_{\max}]$, for all N. Moreover, called $\pi_{N,\min}$ an $\pi_{N,\max}$ respectively the minimum and maximum entries of π_N, we need to assume that there exist two constants c_l and c_u such that $N\pi_{N,\min} \geq c_l$ and $N\pi_{N,\max} \leq c_u$. Notice now that

- by definition of s_N and r_N,

$$s_N = \min_{u,\, v\in V,\, u\neq v}\{d_{\mathrm{E}}(u,\, v)\} \leq \min_{(u,\, v)\in\mathcal{E}}\{d_{\mathrm{E}}(u,\, v)\} \leq r_N$$

  and thus

$$s \leq s_N \leq r_N \leq r;$$

- given a graph $\mathcal{G}_N$, if $\bar{u}$ and $\bar{v}$ are two nodes connected by an edge, we have

$$\rho_N = \min\left\{\frac{d_{\mathrm{E}}(u,\, v)}{d_{\mathcal{G}}(u,\, v)} |\, (u,\, v) \in V \times V\right\} \leq d_{\mathrm{E}}(\bar{u},\, \bar{v}) \leq r_N$$

  and thus

$$\rho \leq \rho_N \leq r;$$

- by definition of γ_N, it is immediate to see that we must have $2\gamma_N \geq s_N$, and thus

$$\frac{s}{2} \leq \gamma_N \leq \gamma.$$

Hence, given the geometric parameters of the family, we have a lower and an upper bound for the actual parameters in any graph. Notice moreover that the degree of each node is bounded by the ratio of the volume of the sphere of radius r_N and the volume of the sphere of radius s_N. Therefore the maximum degree δ_N of the nodes of $\mathcal{G}_N$ is uniformly bounded as follows

$$\delta_N \leq \frac{r_N^d}{s_N^d} \leq \frac{r^d}{s^d}\,.$$

On the other side, clearly for all N we have $\delta_N \geq 1$, because $\mathcal{G}_N$ is connected.

Therefore, similarly to Section 3.3, this discussion allows us to conclude that for the family of reversible consensus matrices P_N constructed as above, it holds

$$k_1 + q_1 f_d(N) \leq J(P_N) \leq k_2 + q_2 f_d(N)$$

where k_1, q_1, k_2 and q_2 depend on d, $p_{\max}$, $p_{\min}$, c_l and c_u and the geometric parameters of the family of graphs s, r, γ, ρ.

5. The relation between the LQ cost and effective resistance: Proofs of Theorem 3.2.1 and Theorem 3.2.2. This section is devoted to the proof of the theorems relating the LQ cost with the average effective resistance. We recall some useful facts from the literature, and then use these notions to prove the results.

5.1. Electrical networks: properties of the effective resistances. This section is devoted to briefly recall without proofs some well-known results on the behavior of the effective resistances in case of perturbation of the electrical network. These are of fundamental importance, since effective resistances show monotonicity properties which are not trivial to prove for consensus matrices without the electrical analogy.

A first important property is the following (see e.g. [24, Thm. B] for a proof).

LEMMA 5.1. *If the electrical network is connected, then the effective resistance is a distance. Namely, it satisfies the following properties:*

- $\mathcal{R}_{uv} \geq 0$ *for all* $u, v \in V$*, and* $\mathcal{R}_{uv} = 0$ *if and only if* $u = v$*;*
- $\mathcal{R}_{uv} = \mathcal{R}_{vu}$*;*
- $\mathcal{R}_{uw} \leq \mathcal{R}_{uv} + \mathcal{R}_{vw}$ *for all* $u, v, w \in V$*.*

A second result, known as Rayleigh's monotonicity law, says that increasing (resp., decreasing) the conductance in any edge of the network implies that the effective resistance between any other couple of nodes respectively cannot increase (resp., decrease). The statement is essentially taken from [18, 25], where the authors were considering a more general case.

LEMMA 5.2 (**Rayleigh's monotonicity law**).
Let C and C' be the conductance matrices of two electrical networks such that

$$C_{uu'} \leq C'_{uu'},\, \forall\, (u,\, u') \in V \times V\,.$$

Then, the effective resistances between any two nodes v, v' in the network are such that

$$\mathcal{R}_{vv'}(C) \geq \mathcal{R}_{vv'}(C').$$

The following lemma [25, Lemma 4.6.1] says that, if we take two resistive networks with the conductance matrices scaled by a constant α, then the effective resistances will be scaled by the constant $1/\alpha$.

LEMMA 5.3.

$$\mathcal{R}_{uu'}(\alpha C) = \frac{1}{\alpha}\mathcal{R}_{uu'}(C),\, \forall\, (u,\, u') \in V \times V.$$

REMARK 5.1.1. *Lemma 5.3 and Rayleigh's monotonicity law imply that the effective resistance in an electrical network is essentially due to the graph topology. In fact,* *if we have an electrical network with conductance matrix C whose non-zero entries belong to the interval $[c_{\min}, c_{\max}]$ and if C' is a conductance matrix having entries equal to 1 in the positions in which C has non-zero entries and to 0 elsewhere, then*

$$\frac{1}{c_{\max}}\mathcal{R}_{uu'}(C') \leq \mathcal{R}_{uu'}(C) \leq \frac{1}{c_{\min}}\mathcal{R}_{uu'}(C'),\ \forall\,(u,\,u') \in V \times V.$$

The last technical lemma deals with h-fuzzing in electrical networks with unitary conductances. Given an integer $h \geq 1$ and a graph $\mathcal{G}$, we call h-fuzz of $\mathcal{G}$, denoted by the symbol $\mathcal{G}^{(h)} = (V^{(h)}, \mathcal{E}^{(h)})$, a graph with the same set of nodes, $V^{(h)} = V$, and with an edge connecting two nodes u and v if and only if the graphical distance $d_{\mathcal{G}}(u,\,v)$ between u and v in $\mathcal{G}$ is at most h, namely

$$\mathcal{E}^{(h)} = \{(u,\,v) \in V \times V :\, d_{\mathcal{G}}(u,\,v) \leq h\}.$$

Notice that, if $h = 1$, then $\mathcal{G}^{(1)} = \mathcal{G}$. If D is the diameter of the graph, namely the maximum graphical distance between a couple of nodes, then $\mathcal{G}^{(D)}$ is the complete graph. It is easy to see that, if P is a stochastic matrix with positive diagonal entries, then the graph $\mathcal{G}_{P^h}$ associated with a P^h is the h-fuzz of the graph $\mathcal{G}_P$ associated with P.

The lemma, which is stated with proof in [25, Lemma 5.5.1], suggests that the effective resistance of $\mathcal{G}$ and of its h-fuzz $\mathcal{G}^{(h)}$ have effective resistances with a similar asymptotic behavior.

Lemma 5.4. *Let $h \in \mathbb{Z}$, $h \geq 1$, and let $\mathcal{G} = (V,\,\mathcal{E})$ be a graph and $\mathcal{G}^{(h)} = (V,\,\mathcal{E}^{(h)})$ be its h-fuzz. For any edge $e \in \mathcal{E}$, define $\mu_h(e)$ to be the number of paths of length at most h passing through e in $\mathcal{G}$ (without any self-loop in the path), and define $\mu_h = \max_{e\in\mathcal{E}} \mu_h(e)$. The following bounds hold true*

$$\frac{1}{h\mu_h}\mathcal{R}_{uv}(\mathcal{G}) \leq \mathcal{R}_{uv}(\mathcal{G}^{(h)}) \leq \mathcal{R}_{uv}(\mathcal{G}).$$

The value of μ_h in the previous result clearly depends on the particular graph under consideration. The following lemma gives a conservative bound for μ_h which depends only on the maximum degree of the nodes in the graph. We will use this bound later, in the particular case of $h = 2$.

Lemma 5.5. *Let μ_h be defined as in Lemma 5.4. If in $\mathcal{G}$ all nodes have degree at most δ, then*

$$\mu_h \leq h^2(\delta - 1)^{h-1}.$$

Proof. For any $K = 1, \dots, h$, we want to find an upper bound on the number of paths of length K passing through the edge e. We let K' be an integer $0 \leq K' \leq K-1$, and we consider the number of paths in which edge e is the $(K'+1)$-th edge in the path, namely there are K' edges before e and $K - K' - 1$ edges after e. As it can be easily seen in Figure 5.1, there are at most $(\delta - 1)^{K'}$ choices for portion of path preceding e, and at most $(\delta-1)^{K-K'-1}$ choices for the portion following e, so that there are at most $(\delta - 1)^{K-1}$ paths having e in $(K'+1)$-th position. Summing upon all $K' = 0, \dots, K-1$, and then summing also upon all path lengths $K = 1, \dots, h$, we obtain that, for any $e \in \mathcal{E}$,

$$\mu_h(e) \leq \sum_{K=1}^{h}\sum_{K'=0}^{K-1}(\delta-1)^{K-1} = \sum_{K=1}^{h} K(\delta-1)^{K-1} \leq \sum_{K=1}^{h} h(\delta-1)^{h-1} = h^2(\delta-1)^{h-1}\,.$$

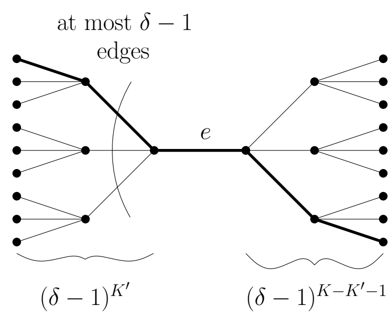

FIGURE 5.1. *Illustration of the proof of Lemma 5.5: upper bound on the number of paths of length K in which e is the $(K'+1)$-th edge, in a graph with node degree at most $\delta = 4$.*

Finally notice that this upper bound for $\mu_h(e)$ is the same for all edges $e \in \mathcal{E}$, and thus it is also an upper bound for the maximum, μ_e. □

5.2. Proof of Theorem 3.2.1 and Theorem 3.2.2. The previous definitions and properties are used in this section to prove the main results Theorem 3.2.1 and Theorem 3.2.2 for reversible consensus matrices. Then Corollary 3.1 and Corollary 3.2 are immediate for symmetric matrices $\pi_i = \frac{1}{N}$ for all $i = 1, \dots, N$. In order to prove the results, we need to introduce two more technical objects which will help us to develop the theory.

Consider a reversible consensus matrix P with invariant measure $\boldsymbol{\pi}$. We call *weighted cost* the following function of P

$$J_{\mathrm{w}}(P) := \operatorname{trace}\left[\sum_{t\geq 0}(I - \boldsymbol{\pi}\mathbf{1}^T)(P^T)^t \Pi P^t (I - \mathbf{1}\boldsymbol{\pi}^T)\right] \tag{5.1}$$

where $\Pi = \operatorname{diag}(\boldsymbol{\pi})$. Notice that in the case of symmetric matrices $J(P) = J_{\mathrm{w}}(P)$. Now, let $C := \Phi(P^2) = N\Pi P^2$. The second object we need is the *weighted average effective resistance*, which is defined as

$$\bar{\mathcal{R}}_{\mathrm{w}}(C) := \frac{1}{2}\boldsymbol{\pi}^T \mathcal{R}(C)\boldsymbol{\pi} = \frac{1}{2}\sum_{(u,\,v)\in V\times V} \mathcal{R}_{uv}(C)\pi_u\pi_v\,. \tag{5.2}$$

Again, notice that in the symmetric case the weighted definition coincides with the un-weighted one, namely $\bar{\mathcal{R}}(C) = \bar{\mathcal{R}}_{\mathrm{w}}(C)$. We present now a lemma which clarifie the relation between the costs $J(P)$ and $J_{\mathrm{w}}(P)$, and between the weighted and the un-weighted average effective resistances, respectively. The proof is immediate from the fact that $\pi_u > 0$ for all u.

LEMMA 5.6. *Let P be a consensus matrix with invariant measure $\boldsymbol{\pi}$ and let $C := \Phi(P^2) = N\Pi P^2$. Then*

$$\frac{1}{N\pi_{\min}} J_{\mathrm{w}}(P) \leq J(P) \leq \frac{1}{N\pi_{\max}} J_{\mathrm{w}}(P)$$

$$\pi_{\min}^2 N^2 \bar{\mathcal{R}}(C) \leq \bar{\mathcal{R}}_{\mathrm{w}}(C) \leq \pi_{\max}^2 N^2 \bar{\mathcal{R}}(C)\ .$$

After the inequalities of the above lemma, which concern separately the LQ cost and the average effective resistance, our goal is to find the relation between the cost of the consensus matrix P and the average effective resistance of the connected electrical network associated with P^2. Before doing so, we need the following technical lemma.

LEMMA 5.7. *If P is a consensus matrix, then the diagonal entries of its Green matrix G are positive.*

Proof. For ease of notation, we prove that $G_{11} > 0$; the proof for the other diagonal entries of G can be obtained by the same arguments. We fix the following notation: we let $\boldsymbol{g}^T = [G_{11}, \tilde{\boldsymbol{g}}^T]$ be the first row of G, and we define the following partitions

$$L = \begin{bmatrix} l_{11} & \boldsymbol{r}_1^T \\ \boldsymbol{c}_1 & \tilde{L} \end{bmatrix}, \; P = \begin{bmatrix} p_{11} & {\boldsymbol{r}'}_1^T \\ \boldsymbol{c}'_1 & \tilde{P} \end{bmatrix}, \; \boldsymbol{g}^T = \begin{bmatrix} G_{11}, \tilde{\boldsymbol{g}}^T \end{bmatrix}, \; \boldsymbol{\pi}^T = \begin{bmatrix} \pi_1, \tilde{\boldsymbol{\pi}}^T \end{bmatrix}.$$

Because $GL = I - P$, we have $\boldsymbol{g}^T L = \boldsymbol{e}_1^T - \boldsymbol{\pi}^T$, where $\boldsymbol{e}_1$ denotes the first vector of the canonical basis of $\mathcal{R}^N$. Notice that $G\mathbf{1} = \mathbf{0}$ implies that $\boldsymbol{g}^T\mathbf{1} = 0$ and thus, in particular, $G_{11} = -\tilde{\boldsymbol{g}}^T\mathbf{1}_{N-1}$. Similarly, $\boldsymbol{\pi}^T L = \mathbf{0}^T$ gives $l_{11} = -\frac{1}{\pi_1}\tilde{\boldsymbol{\pi}}^T\boldsymbol{c}_1$. Hence, we can write the equality $\boldsymbol{g}^T L = \boldsymbol{e}_1^T - \boldsymbol{\pi}^T$ in the following equivalent way

$$\tilde{\boldsymbol{g}}_1^T \begin{bmatrix} -\mathbf{1}_{N-1} & I_{N-1} \end{bmatrix} \begin{bmatrix} -\frac{1}{\pi_1}\tilde{\boldsymbol{\pi}}^T \\ I_{N-1} \end{bmatrix} \tilde{L} \begin{bmatrix} -\mathbf{1}_{N-1} & I_{N-1} \end{bmatrix} = \boldsymbol{e}_1^T - \boldsymbol{\pi}^T.$$

By right-multiplying both sides of the above equality with a factor $\begin{bmatrix} \mathbf{0}_{N-1}^T \\ \tilde{L}^{-1} \end{bmatrix} \mathbf{1}_{N-1}$, we obtain that $\tilde{\boldsymbol{g}}_1^T\mathbf{1}_{N-1} = -\tilde{\boldsymbol{\pi}}^T\tilde{L}^{-1}\mathbf{1}_{N-1}$, and thus

$$G_{11} = \tilde{\boldsymbol{\pi}}^T\tilde{L}^{-1}\mathbf{1}_{N-1}.$$

Now notice that, by the definition $L = I - P$ of the Laplacian, $\tilde{L} = I - \tilde{P}$. Moreover, our definition of consensus matrix implies that P is primitive, and thus it is well-known that $\tilde{P}$ has all eigenvalues inside the unit circle (see e.g. [26, Lemma III.1] for a proof). This implies that $\tilde{L}$ is invertible, and that the series $\sum_{t\geq 0}\tilde{P}^t$ is convergent and is equal to $(I-\tilde{P})^{-1} = \tilde{L}^{-1}$. This allows to obtain

$$G_{11} = \tilde{\boldsymbol{\pi}}^T \sum_{t\geq 0} \tilde{P}^t \mathbf{1}_{N-1}.$$

Recalling that the entries of $\tilde{\boldsymbol{\pi}}$ are all positive, and that $\tilde{P}$ has non-negative entries with at least some positive element, this proves that $G_{11} > 0$. □

Now we have the tools to prove the following lemma, which shows the relation between the weighted cost and the weighted average effective resistance.

LEMMA 5.8. *Let P be a reversible consensus matrix with invariant measure $\boldsymbol{\pi}$ and let $C := \Phi(P^2) = N\Pi P^2$. Then*

$$\pi_{\min} N\bar{\mathcal{R}}_{\mathrm{w}}(C) \leq J_{\mathrm{w}}(P) \leq \pi_{\max} N\bar{\mathcal{R}}_{\mathrm{w}}(C).$$

Proof. To prove this lemma, we will prove the following two equalities involving the Green matrix associated with P^2, which we will denote by $G(P^2)$

1. $J_{\mathrm{w}}(P) = \operatorname{trace} \Pi G(P^2)$;

2. $\bar{\mathcal{R}}_{\mathrm{w}}(C) = \frac{1}{N}\operatorname{trace} G(P^2)$.

From such equalities, the statement follows, because Π is diagonal and positive definite, and $G(P^2)$ has positive diagonal entries (see Lemma 5.7).

As far as the first equality is concerned, observe that

$$\begin{aligned} J_{\mathrm{w}}(P) &= \operatorname{trace}\left(\sum_{t\geq 0}(P^t - \mathbf{1}\boldsymbol{\pi}^T)^T\Pi(P^t - \mathbf{1}\boldsymbol{\pi}^T)\right) \\ &= \operatorname{trace}\left(\sum_{t\geq 0}\left((P^t)^T\Pi P^t - \boldsymbol{\pi}\boldsymbol{\pi}^T\right)\right) \\ &= \operatorname{trace}\left(\sum_{t\geq 0}\Pi\left(P^{2t} - \mathbf{1}\boldsymbol{\pi}^T\right)\right) = \operatorname{trace}\left(\Pi G(P^2)\right). \end{aligned}$$

As far as the second equality is concerned, observe that, by substituting the expression for $\mathcal{R}_{uv}(C)$ given in Eq. (3.8) inside the definition of $\bar{\mathcal{R}}_{\mathrm{w}}(C)$, we get

$$\bar{\mathcal{R}}_{\mathrm{w}}(C) = \frac{1}{2}\sum_{u,v}\tfrac{1}{N}(\boldsymbol{e}_u - \boldsymbol{e}_v)^T G(P^2)\Pi^{-1}(\boldsymbol{e}_u - \boldsymbol{e}_v)\pi_u\pi_v,$$

from which we can compute

$$\begin{aligned} \bar{\mathcal{R}}_{\mathrm{w}}(C) &= \frac{1}{2}\sum_{u,v}\tfrac{1}{N}(\boldsymbol{e}_u - \boldsymbol{e}_v)^T G(P^2)\Pi^{-1}(\boldsymbol{e}_u - \boldsymbol{e}_v)\pi_u\pi_v \\ &= \frac{1}{N}\frac{1}{2}\sum_{u,v}(\boldsymbol{e}_u^T - \boldsymbol{e}_v^T)G(P^2)(\pi_v\boldsymbol{e}_u - \pi_u\boldsymbol{e}_v) \\ &= \frac{1}{N}\Big(\frac{1}{2}\sum_{u,v}(\pi_v\boldsymbol{e}_u^T G(P^2)\boldsymbol{e}_u + \pi_u\boldsymbol{e}_v^T G(P^2)\boldsymbol{e}_v) \\ &\qquad - \frac{1}{2}\sum_{u,v}(\pi_v\boldsymbol{e}_v^T G(P^2)\boldsymbol{e}_u + \pi_u\boldsymbol{e}_u^T G(P^2)\boldsymbol{e}_v)\Big) \\ &= \frac{1}{N}\left(\operatorname{trace}(G(P^2)) - \boldsymbol{\pi}^T G(P^2)\mathbf{1}\right), \end{aligned}$$

which yields the proof of the equality since $\boldsymbol{\pi}^T G(P^2)\mathbf{1} = 0$. □

These lemmas can be easily used to infer the first main result, since Theorem 3.2.1's proof follows immediately from the inequalities in Lemma 5.6 together with those in Lemma 5.8.

In order to prove the second main result, we need a last technical lemma, which allows us to reduce the computation of the average effective resistances on the 2-fuzz of $\mathcal{G}$ to those on $\mathcal{G}$ only.

LEMMA 5.9. *Let P be a reversible consensus matrix with invariant measure $\boldsymbol{\pi}$ and with associated graph $\mathcal{G}$. Let $C := \Phi(P^2)$. Then*

$$\frac{1}{8N\pi_{\max}\delta^2 p_{\max}^2}\bar{\mathcal{R}}(\mathcal{G}) \leq \bar{\mathcal{R}}(C) \leq \frac{1}{N\pi_{\min}p_{\min}^2}\bar{\mathcal{R}}(\mathcal{G}), \tag{5.3}$$

where δ denotes the largest degree of the graph nodes in $\mathcal{G}$ and $p_{\min}$ and $p_{\max}$ are, respectively, the minimum and the maximum of the non-zero elements of P.

Proof. First of all notice that, for all u, v such that $C_{uv} \neq 0$ we have that

$$C_{uv} = N\pi_u [P^2]_{uv} = N\pi_u \sum_w P_{uw}P_{wv}.$$

By definition of $p_{\min}$ and $p_{\max}$, and because there are at most $\delta + 1$ non-zero terms P_{uw} for any fixed u, this yields

$$\forall C_{uv} \neq 0,\ N\pi_{\min} p_{\min}^2 \leq C_{uv} \leq N\pi_{\max}(\delta+1)p_{\max}^2. \tag{5.4}$$

By Remark 5.1.1, $\frac{1}{c_{\max}}\bar{\mathcal{R}}(\mathcal{G}^{(2)}) \leq \bar{\mathcal{R}}(C) \leq \frac{1}{c_{\min}}\bar{\mathcal{R}}(\mathcal{G}^{(2)})$, where $c_{\min}$ and $c_{\max}$ are the minimum and maximum non-zero entries of C, respectively. This, together with Eq. (5.4), gives

$$\frac{1}{N\pi_{\max}(\delta+1)p_{\max}^2}\bar{\mathcal{R}}(\mathcal{G}^{(2)}) \leq \bar{\mathcal{R}}(C) \leq \frac{1}{N\pi_{\min}p_{\min}^2}\bar{\mathcal{R}}(\mathcal{G}^{(2)})\,.$$

Then we apply Lemmas 5.4 and 5.5, both with $h = 2$, and we obtain

$$\frac{1}{8(\delta-1)N\pi_{\max}(\delta+1)p_{\max}^2}\bar{\mathcal{R}}(\mathcal{G}) \leq \bar{\mathcal{R}}(C) \leq \frac{1}{N\pi_{\min}p_{\min}^2}\bar{\mathcal{R}}(\mathcal{G})\,,$$

which yields the claim, because $(\delta+1)(\delta-1) \leq \delta^2$. ▯

Now we can prove the second main result: Theorem 3.2.2 immediately follows from Theorem 3.2.1 and Lemma 5.9.

6. Application to geometric graphs: Proof of Theorem 4.1.1. In order to prove Theorem 4.1.1, we need two preliminary results. The first one is an immediate corollary of a theorem taken from [6], which states that the claimed asymptotic behavior of geometric graphs holds true at least in the case of regular grids. Recall that, by regular grid, we mean a d-dimensional geometric graph with $N = n^d$ nodes lying on the points $(i_1, \dots, i_d)$, where $i_1, \dots, i_d \in \{0, \dots, n-1\}$ and in which there is an edge connecting two nodes u, v if and only if their distance is $d_{\mathrm{E}}(u, v) \leq 1$.

LEMMA 6.1. *Let $B_{\mathcal{L}}$ be the incidence matrix of a regular grid in dimension d with $N = n^d$ nodes. Let $P \in \mathbb{R}^{N\times N}$ be the consensus matrix defined as follows*

$$P = I - \frac{1}{2d+1}B_{\mathcal{L}}^T B_{\mathcal{L}}$$

whose associated graph is the regular grid. Then

$$c_l f_d(N) \leq \bar{\mathcal{R}}(\mathcal{L}) \leq c_u f_d(N)$$

where c_l and c_u depend on d only, and where $f_d(N)$ is defined in Eq. (4.5).

Proof. With the same assumptions, from [6], Proposition 1, we known that

$$c_l' f_d(N) \leq J(P) \leq c_u' f_d(N)\,,$$

where c_l' and c_u' depend on δ only. The result immediately follows from Corollary 3.2. ▯

The second result allows us to reduce the problem of computing the average effective resistance in the geometric graph to the simpler case of two suitable grids. First of all, we state the following three technical results.

LEMMA 6.2. *In an hypercube $H \subseteq Q$ with side length less than $\frac{s}{\sqrt{d}}$, there is at most one node $u \in \mathcal{V}$. In an hypercube $H' \subseteq Q$ with side length greater than 2γ, there is at least one node $u' \in \mathcal{V}$.*

Proof. If the side length of an hypercube is $\frac{s}{\sqrt{d}}$, then its diagonal has length s. If we had two nodes in the hypercube, their distance would be less than s, in contradiction with the definition of s. The second claim is proved noticing that an hypercube of side length 2γ includes a sphere of radius γ. If it did not contain any node, then we could find a sphere of radius larger than γ not containing any node, in contradiction with the definition of γ. ◻

As a corollary of the previous lemma we have the following result.

LEMMA 6.3. *Let H be a hypercube in Q with edge length ℓ_H and let N_H be the number of nodes in it. Then*

$$\left\lfloor \frac{\ell_H}{2\gamma} \right\rfloor < \sqrt[d]{N_H} < \left\lceil \frac{\sqrt{d}\ell_H}{s} \right\rceil .$$

Proof. The result follows from Lemma 6.2 simply counting how many disjoint hypercubes of side length $\frac{s}{\sqrt{d}}$ and 2γ we can find in an hypercube of side length ℓ_H. ◻

In particular for the whole graph we have the following corollary.

COROLLARY 6.4. *The number of nodes N of the graph is such that*

$$\ell \frac{1-\frac{2\gamma}{\ell}}{2\gamma} < \sqrt[d]{N} < \ell \frac{\sqrt{d}-\frac{s}{\ell}}{s}.$$

Notice that, in the case where ℓ is big with respect to γ and s, the previous corollary essentially implies that N is proportional to ℓ^d. The following lemma concerns geometric graphs and their embeddings in lattices.

LEMMA 6.5. *Let $\mathcal{G} = (V, \mathcal{E})$ be a geometric graph with parameters (s, r, γ, ρ) and with nodes in an hypercube $Q = [0, \ell]^d$ in which $\gamma < \frac{\ell}{4}$. Then there exist two lattices, $\mathcal{L}_1$ and $\mathcal{L}_2$ such that*

$$k_1 + q_1 \bar{\mathcal{R}}(\mathcal{L}_1) \le \bar{\mathcal{R}}(\mathcal{G}) \le k_2 + q_2 \bar{\mathcal{R}}(\mathcal{L}_2), \tag{6.1}$$

where q_1, q_2, k_1 and k_2 depend on s, r, γ, ρ, and on d. Moreover, there exist four constants, c_1', c_1'', c_2', and c_2'', depending on the same set of parameters, such that, if N_1 and N_2 are respectively the number of nodes of $\mathcal{L}_1$ and $\mathcal{L}_2$, then

$$c_1' N_1 \le N \le c_1'' N_1 \qquad c_2' N_2 \le N \le c_2'' N_2. \tag{6.2}$$

Proof. The idea is to tessellate the hypercube Q in order to obtain a rough approximation of $\mathcal{G}$, and then compute the bound for the effective resistance. Let us consider the upper bound first. Define $n_1 := \lceil \frac{\ell}{2\gamma} \rceil - 1$ and $\lambda := \frac{\ell}{n_1}$ and (exactly) tessellate the hypercube Q with $N_1 := n_1^d$ hypercubes of side length λ as in Fig. 6.1. Notice that the technical assumption $\gamma < \frac{\ell}{4}$ also implies $\gamma < \frac{\ell}{2}$, which in turn avoids the pathological case in which $n_1 = 0$. Using the properties of $\lceil \cdot \rceil$, it can be seen that

$$2\gamma < \lambda < 4\gamma.$$

Notice that the assumption $\ell > 4\gamma$ ensures that $N_1 \ge 2^d$.

Notice that, by Lemma 6.2, in each of these hypercubes there is at least one node of the graph $\mathcal{G}$. On the other hand, by Lemma 6.3, we can argue that in each hypercube there are at most $\lceil \frac{\sqrt{d}\lambda}{s} \rceil^d$ nodes. Since $2\gamma < \lambda$, then

$$\frac{1}{\lceil \frac{2\sqrt{d}\gamma}{s} \rceil^d} N \le N_1 \le N.$$

This proves the first of the two bounds in Eq. 6.2. Another consequence of the fact that in each of these hypercubes there is at least one node of $\mathcal{G}$ is that for each hypercube we can select one "representative" node in V belonging to it. Let $V_{\mathcal{L}_1} \subseteq V$ be the set of these representatives. Consider now the regular lattice $\mathcal{L}_1 = (V_{\mathcal{L}_1}, \mathcal{E}_{\mathcal{L}_1})$ having as the set of nodes the set of representatives $V_{\mathcal{L}_1}$ and in which there exists an edge connecting two nodes in $V_{\mathcal{L}_1}$ if the two corresponding hypercubes touch each other (not diagonally). Define the function $\eta : V \to V_{\mathcal{L}_1}$ such that $\eta(u) = u'$ if u belongs to the hypercube associated with u'.

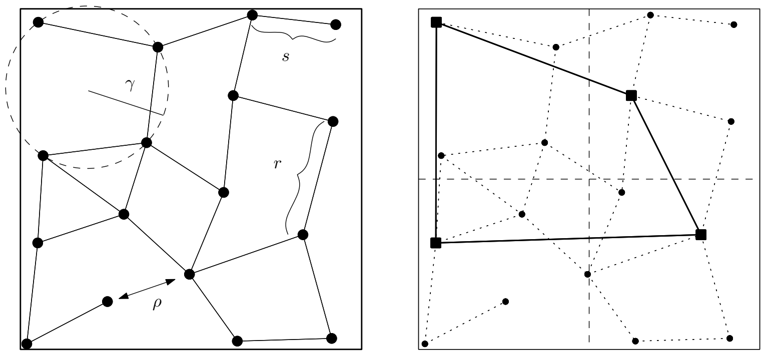

FIGURE 6.1. *On the left, an example of geometric graph in $\mathbb{R}^2$ with parameters s, r, γ and ρ (for ρ, the two nodes for which the minimum in the definition is attained). On the right, the lattice $\mathcal{L}_1$ built for the upper bound. The box-marked nodes are the representatives of the hypercubes, in thick solid line the edges of the lattice $\mathcal{L}_1$. Small nodes and dotted lines are the other nodes and edge of the original graph $\mathcal{G}$.*

The next step is to prove that there exists an integer $h \geq 1$ such that the h-fuzz $\mathcal{G}^{(h)}$ of $\mathcal{G}$ embeds $\mathcal{L}_1$, namely that all the nodes and edges of $\mathcal{L}_1$ are also nodes and edges of $\mathcal{G}^{(h)}$. Take thus $u', v' \in V_{\mathcal{L}_1}$ such that $(u', v') \in \mathcal{E}_{\mathcal{L}_1}$. Their Euclidean distance is bounded as follows

$$d_{\mathrm{E}}(u', v') \leq \lambda\sqrt{d+3}$$

as a simple geometric argument shows. By definition of ρ, we obtain

$$d_{\mathcal{G}}(u',\, v') \leq \frac{\lambda\sqrt{d+3}}{\rho} \leq \frac{4\gamma\sqrt{d+3}}{\rho}.$$

Take thus $h = \lfloor \frac{4\gamma\sqrt{d+3}}{\rho} \rfloor$ and build $\mathcal{G}^{(h)}$. By the previous discussion, it is manifest that $\mathcal{G}^{(h)}$ embeds $\mathcal{L}_1$.

Now, we claim that in $\mathcal{G}^{(h)}$ all the nodes lying in the same hypercube have graphical distance 1, namely they are all connected each other. In fact, if u and v lie in one hypercube, and thus $d_{\mathrm{E}}(u,\, v) \leq \lambda\sqrt{d}$, then we have

$$d_{\mathcal{G}}(v,\, u) \leq \frac{1}{\rho} d_{\mathrm{E}}(v,\, u) \leq \frac{\lambda\sqrt{d}}{\rho} \leq \frac{4\gamma\sqrt{d}}{\rho} \leq \frac{4\gamma\sqrt{d+3}}{\rho},$$

and thus $d_{\mathcal{G}}(v,\, u) \leq h$. This clearly yields

$$d_{\mathcal{G}^{(h)}}(u,\, \eta(u)) \leq 1. \tag{6.3}$$

We can now prove the claim. Since $\mathcal{G}^{(h)}$ embeds $\mathcal{L}_1$, by the properties of the effective resistances, for each $u', v' \in V_{\mathcal{L}_1}$ we have that

$$\mathcal{R}_{u'v'}(\mathcal{G}^{(h)}) \leq \mathcal{R}_{u'v'}(\mathcal{L}_1).$$

This is still limited to the set of representatives $V_{\mathcal{L}_1}$. If u and v are two generic nodes of $\mathcal{G}^{(h)}$, using Eq. (6.3) and the fact that the effective resistance is a distance (Lemma 5.1), we can obtain that

$$\begin{aligned}\mathcal{R}_{u,\,v}(\mathcal{G}^{(h)}) &\leq \mathcal{R}_{u,\,\eta(u)}(\mathcal{G}^{(h)}) + \mathcal{R}_{\eta(u),\,\eta(v)}(\mathcal{G}^{(h)}) + \mathcal{R}_{\eta(v),\,v}(\mathcal{G}^{(h)}) \\ &\leq 2 + \mathcal{R}_{\eta(u),\,\eta(v)}(\mathcal{G}^{(h)}).\end{aligned}$$

Thus, we have

$$\begin{aligned}\bar{\mathcal{R}}(\mathcal{G}^{(h)}) &= \frac{1}{2N^2}\sum_{u,v\in V}\mathcal{R}_{u,\,v}(\mathcal{G}^{(h)}) \leq 1 + \frac{1}{2N^2}\sum_{u,v\in V}\mathcal{R}_{\eta(u),\,\eta(v)}(\mathcal{G}^{(h)}) \\ &\leq 1 + \frac{1}{2N^2}\sum_{u,v\in V}\mathcal{R}_{\eta(u),\,\eta(v)}(\mathcal{L}_1) = 1 + \frac{1}{2N^2}\sum_{u',v'\in V_{\mathcal{L}_1}}\sum_{\substack{u\in\eta^{-1}(u')\\ v\in\eta^{-1}(v')}}\mathcal{R}_{u',v'}(\mathcal{L}_1) \\ &\leq 1 + \frac{M^2}{2N^2}\sum_{u',v'\in V_{\mathcal{L}_1}}\mathcal{R}_{u',v'}(\mathcal{L}_1) = 1 + M^2\frac{N_1^2}{N^2}\bar{\mathcal{R}}(\mathcal{L}_1)\end{aligned}$$

where, as already pointed out, M, the maximum number of nodes of $\mathcal{G}$ in each hypercube of length λ, can be bounded as $M \leq \lceil \frac{\sqrt{d}\lambda}{s} \rceil^d$. By previous arguments, $M\frac{N_1}{N}$ can be bounded from above by a constant dependent on the geometric parameters of the geometric graph and on d. Thus, the claim of the Lemma immediately descends from Lemma 5.4.

The proof for the lower bound follows basically the same steps once a good regular lattice candidate is selected. We tessellate again Q by means of hypercubes of side length

$$\lambda := \frac{\ell}{\lfloor \frac{\ell\sqrt{d}}{s} \rfloor + 1}$$

as in Fig. 6.2. Observe that $\lambda < s/\sqrt{d}$ so that by Lemma 6.2 in each of them there can be at most one node. The candidate lattice is $\mathcal{L}_2 = (V_{\mathcal{L}_2}, \mathcal{E}_{\mathcal{L}_2})$, where $V_{\mathcal{L}_2}$ is the set of hypercubes and the edges connect again two nodes in $V_{\mathcal{L}_2}$ if the two corresponding hypercubes touch each other (not diagonally).

It can be proved that, if we take $u, v \in V$ such that $(u, v) \in \mathcal{E}$, then $d_{\mathcal{L}_2}(u, v) \leq d\lceil r/\lambda \rceil$. We define now the map $\eta : V_{\mathcal{L}_2} \to V$ so that $\eta(u')$ is the node in V which is closest to u' in the Euclidean distance. It can be proved that, for all $u' \in V_{\mathcal{L}_2}$ we have that $d_{\mathrm{E}}(u', \eta(u')) \leq \gamma$ and so for any pair of nodes u' and v' such that $\eta(u') = \eta(v')$ we have that $d_{\mathrm{E}}(u', v') \leq 2\gamma$ and consequently $d_{\mathcal{L}_2}(u', v') \leq d\lceil 2\gamma/\lambda \rceil$.

Analogously to the upper bound case, we write $V \subseteq V_{\mathcal{L}_2}$ identifying a node of the graph with the hypercube it belongs to. Once this is done, and taking $h := \max\{d\lceil r/\lambda \rceil, d\lceil 2\gamma/\lambda \rceil\}$ we can argue that $\mathcal{L}_2^{(h')}$ embeds $\mathcal{G}$ and that, given $u \in V$, for any pair of nodes u' and v' in $\eta^{-1}(u)$ we have

$$d_{\mathcal{L}_2^{(h)}}(u', v') \leq 1.$$

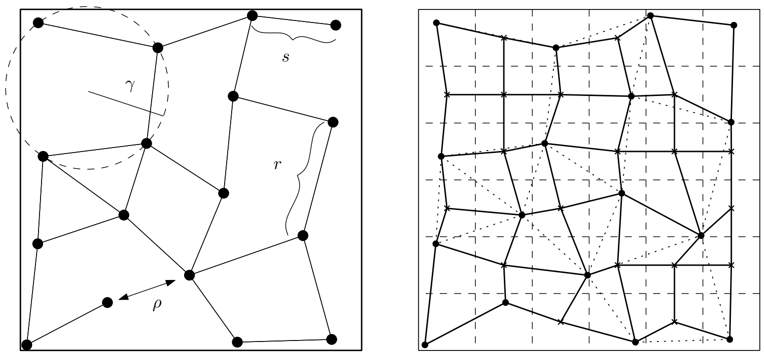

Figure 6.2. *On the left, the geometric graph already used for the upper bound. On the right, the lattice $\mathcal{L}_2$ built for the lower bound. The centers of the hypercubes in which there are no nodes of $\mathcal{G}$ are marked by a cross, while the bullet nodes are the nodes of $\mathcal{G}$. In solid lines are all the edges of $\mathcal{L}_2$, in dotted lines the other edges of the original graph $\mathcal{G}$.*

The last part of the proof, including the second bound in Eq. 6.2, is totally analogous to the upper bound case. ∎

We can now prove our theorem on geometric graphs.

Proof. [of Theorem 4.1.1] We know by Theorem 3.2.2 that

$$c_l \bar{\mathcal{R}}(\mathcal{G}) \leq J(P) \leq c_u \bar{\mathcal{R}}(\mathcal{G})$$

with c_l and c_u dependent on $p_{\min}$, $p_{\max}$, δ and the products $\pi_{\min} N$ and $\pi_{\max} N$. By Lemma 6.5, we can argue that

$$k_1' + q_1' \bar{\mathcal{R}}(\mathcal{L}_1) \leq J(P) \leq k_2' + q_1' \bar{\mathcal{R}}(\mathcal{L}_2) \tag{6.4}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two lattices such that $c_1' N_1 \leq N \leq c_1'' N_1$, $c_2' N_2 \leq N \leq c_2'' N_2$ and where k_1', q_1', k_2' and q_2' is a set of constants dependent on $p_{\min}$, $p_{\max}$, δ, the products $\pi_{\min} N$ and $\pi_{\max} N$, d and the parameters of the geometric graph.

Take now the grid $\mathcal{L}_1$, let $B_{\mathcal{L}_1}$ be its adjacency matrix, and build the consensus matrix

$$P_1 = I - \frac{1}{2d+1} B_{\mathcal{L}_1}^T B_{\mathcal{L}_1}.$$

By Lemma 6.1, we know that

$$\alpha_1 f_d(N_1) \leq \bar{\mathcal{R}}(\mathcal{L}_1) \leq \alpha_2' f_d(N_1),$$

where α_1 and α_2 depend on the parameter δ only.

Notice now that Lemma 6.5 also states that $c_1' N_1 \leq N \leq c_1'' N_1$, where c_1' and c_1'' depend on the parameters of the geometric graph only. Simple computations using the definition of $f_d(\cdot)$ in Eq. (4.5) yield to

$$k_1' + q_1' f_d(N) \leq \bar{\mathcal{R}}(\mathcal{L}_1) \leq k_1'' + q_1'' f_d(N),$$

where k_1', q_1', k_1'' and q_1'' depend on the geometric parameters and on d.

Analogously, there exists a symmetric consensus matrix P_2 associated with $\mathcal{L}_2$ for which

$$k_2' + q_2' f_d(N) \leq \bar{\mathcal{R}}(\mathcal{L}_1) \leq k_2'' + q_2'' f_d(N),$$

where k_2', q_2', k_2'' and q_2'' depend on the geometric parameters and on d.

By substituting in Eq. (6.4), it is now clear that

$$k_1 + q_1 f_d(N) \leq J(P) \leq k_2 + q_2 f_d(N)$$

with k_1, q_1, k_2 and q_2 as in the statement of the theorem. □

7. Conclusion. In this paper we study an LQ cost which measures the performance of a consensus algorithm. We show that, under mild assumptions on the associated communication graph, if the consensus matrix is reversible then the LQ cost can be bounded using a quantity related to the graph topology only, namely its average effective resistance. For the generic reversible matrix a strong condition on the entries of the invariant measure must be satisfied in order the bound to be effective. However, we have shown that some highly popular and easily implementable strategies implicitly fulfil it. Exploiting this result, we study a large class of graphs, called geometric graphs, which describe geometrically local communication and which can be seen as perturbed grids. We show the the cost exhibits a particular behaviour as a function of the number of nodes in the graph, which is related to the dimension of the space in which the graph is drawn only, extending a result already known for highly symmetric graphs.

REFERENCES

[1] G. Cybenko. Dynamic load balancing for distributed memory multiprocessors. *J. Parallel Distrib. Comput.*, 7(2):279–301, 1989.

[2] J. Cortes, S. Martinez, and F. Bullo. Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Transactions on Automatic Control*, 51(8):1289–1298, 2006.

[3] R. Olfati-Saber. Distributed Kalman filter with embedded consensus filters. In *Proceedings of 44th IEEE Conference on Decision and Control and European Control Conference CDC-ECC'05*, pages 8179–8184, dec. 2005.

[4] S. Ganeriwal, R. Kumar, and M.B. Srivastava. Timing-sync protocol for sensor networks. In *Proceedings of the 1st international conference on Embedded networked sensor systems (SenSys '03)*, pages 138–149, 2003.

[5] D.A. Levin, Y. Peres, and E.L. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, 2008.

[6] R. Carli, F. Garin, and S. Zampieri. Quadratic indices for the analysis of consensus algorithms. *ITA Workshop 2009*, 2009.

[7] B. Bamieh, M. Jovanovic, P. Mitra, and S. Patterson. Coherence in large-scale networks: Dimension dependent limitations of local feedback. *IEEE Transactions on Automatic Control (submitted)*, 2009.

[8] F. Garin and L. Schenato. *Distributed estimation and control applications using linear consensus algorithms*, volume Networked control systems, A. Bemporad, M. Heemels, M. Johansson eds. of *Lecture Notes in Control and Information Sciences*, chapter 3. Springer, 2011.

[9] F. Garin and S. Zampieri. Mean square performance of consensus-based distributed estimation over regular geometric graphs. *Siam Journal Control and Optimization*, 2011. to appear.

[10] L. Xiao, S. Boyd, and S.-J. Kimb. Distributed average consensus with least-mean-square deviation. *Journal of Parallel and Distributed Computing*, 2007.

[11] P.G. Doyle and J.L. Snell. *Random Walks and Electric Networks.* Carus Monographs. Mathematical Association of America, 1984.

[12] E. Lovisari, F. Garin, and S. Zampieri. A resistance-based approach to performance analysis of the consensus algorithm. In *Proceedings of the 44th Conference on Decision and Control CDC 2010*, pages 5714–5719, Dec. 2010.

[13] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *Information Theory, IEEE Transactions on*, 52(6):2508 – 2530, june 2006.

[14] E. Lovisari and S. Zampieri. Performance metrics in the average consensus problem: a tutorial. *Annual Reviews in Control (to appear)*, 2012.

[15] P. Diaconis and D. Stroock. Geometric bounds for eigenvalues of Markov chains. *The Annals of Applied Probability*, 1(1):36–61, 1991.
[16] D. Aldous and J. Fill. *Reversible Markov Chains and Random Walks on Graphs.* . Draft available online at `http://www.stat.berkeley.edu/~aldous/RWG/book.html`.
[17] P. Barooah and J. Hespanha. Estimation from relative measurements: Electrical analogy and large graphs. *IEEE Transactions on Signal Processing*, 56(6):2181–2193, june 2008.
[18] P. Barooah and J. Hespanha. Estimation from relative measurements: Error bounds from electrical analogy. *ICISIP*, 2005.
[19] P. Barooah and J. Hespanha. Error scaling laws for linear optimal estimtation from relative measurements. *IEEE Transactions on Information Theory*, 55(12):5661–5673, dec. 2009.
[20] A. Giridhar. *In-Network Computation in Wireless Sensor Networks*. PhD thesis, University of Illinois, Urbana-Champaign, 2006.
[21] A. Giridhar and P.R. Kumar. Distributed clock synchronization over wireless networks: Algorithms and analysis. In *Proceedings of the 45th IEEE Conference on Decision and Control CDC'06*, pages 4915–4920, Dec. 2006.
[22] A. Ghosh, S. Boyd, and A. Saberi. Minimizing effective resistance of a graph. *SIAM review*, 50(1):37–66, 2008.
[23] F.Y. Wu. Theory of resistor networks: the two-point resistance. *J. Phys. A: Math. Gen.*, 2004.
[24] D.J. Klein and M. Randić. Resistance distance. *Journal of Mathematical Chemistry*, 12:81–95, 1993.
[25] P. Barooah. *Estimation and Control with Relative Measurements: Algorithms and Scaling Laws*. PhD thesis, University of California, Santa Barbara, 2007.
[26] F. Pasqualetti, A. Bicchi, and F. Bullo. Consensus computation in unreliable networks: A system theoretic approach. 56(12), 2011. To appear. Draft available online at `http://motion.me.ucsb.edu/pdf/2009b-pbb.pdf`.