



Lecture 6 and 7

Lecture 6

- ▶ Linear Programming (LP)
- ▶ LP in production planning example
- ▶ Model Predictive Control
- ▶ A portfolio optimization problem

Lecture 7

- ▶ More general convex optimization
- ▶ Portfolio optimization revisited
- ▶ Duality and distributed optimization

Mini Problem

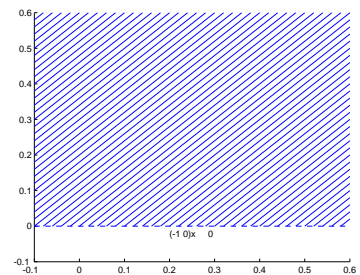
$$\begin{aligned} \text{Minimize} \quad & -x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 1 \\ & 2x_1 + x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Equivalent matrix formulation:

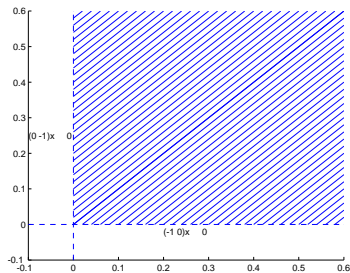
$$\begin{aligned} \text{Minimize} \quad & \begin{pmatrix} -1 & -1 \end{pmatrix} x \\ \text{subject to} \quad & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x \preceq \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x \succeq 0 \end{aligned}$$

where $x = (x_1 \ x_2)^T$

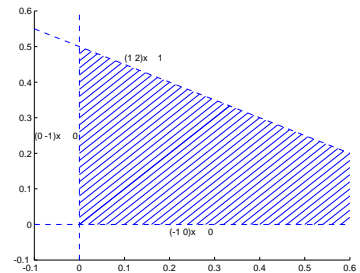
Mini Problem graphical solution



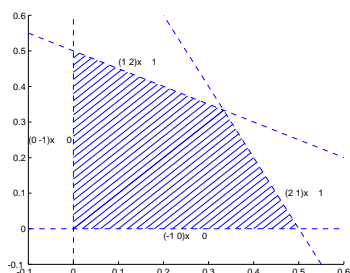
Mini Problem graphical solution



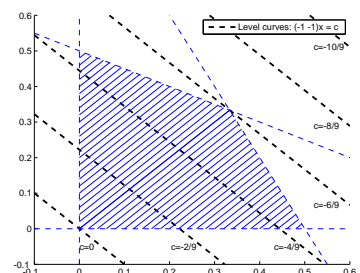
Mini Problem graphical solution



Mini Problem graphical solution



Mini Problem graphical solution



Linear Programming

General formulation:

$$\begin{aligned} &\text{Minimize} && c^T x \\ &\text{subject to} && Ax \preceq b \\ &&& Hx = g \end{aligned}$$

Today's lecture

- ▶ Linear Programming (LP)
- ▶ LP in production planning example
 - ▶ Static systems
 - ▶ Dynamical systems
- ▶ Model Predictive Control
- ▶ A Portfolio Optimization Problem

Production planning example

Two products are produced:

- ▶ Garden furniture
- ▶ Sleds

Two main parts of production

- ▶ Sawing
- ▶ Assembling

Production planning example cont'd

Weekly production:

x_1 : Garden furniture

x_2 : Sleds

Product prices:

p_1 : Garden furniture

p_2 : Sleds

The objective is to maximize weekly profit:

$$\max p_1 x_1 + p_2 x_2$$

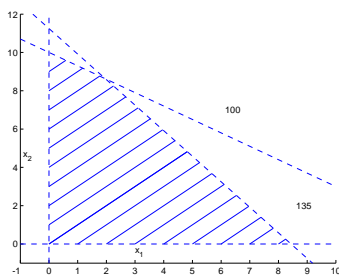
Subject to:

Sawing constraints: $7x_1 + 10x_2 \leq 100$

Assembling constraints: $16x_1 + 12x_2 \leq 135$

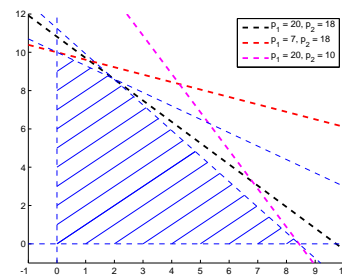
Production planning example cont'd

Sawing and assembling constraints:



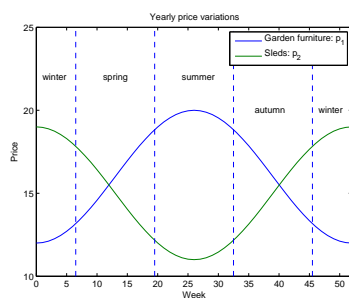
Production planning example cont'd

Level curves for optimal points obtained with different prices:



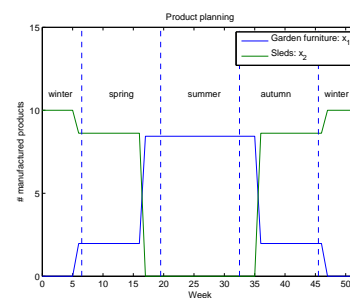
Production planning example cont'd

Seasonal variations in expected prices:



Production planning example cont'd

Optimal production for different seasons:



Today's lecture

- Linear Programming (LP)
- LP in production planning example
 - Static systems
 - Dynamical systems
- Model Predictive Control
- A portfolio optimization problem

Dynamic Production planning example

Hire extra personnel to increase production:

Nominal learning (sawing):

$$x_3(t+1) = 0.7x_3(t) + 30u_3(t)$$

Nominal learning (assembling):

$$x_4(t+1) = 0.7x_4(t) + 40.5u_4(t)$$

where $u_3 \in [0, 1]$, $u_4 \in [0, 1]$ is fraction of full time employment

$x_3(t)$ and $x_4(t)$ quantifies increased capacity:

Sawing: $7x_1 + 10x_2 \leq 100 + x_3(t)$

Assembling: $16x_1 + 12x_2 \leq 135 + x_4(t)$

Mini problem

Assume that extra sawing personnel is working full-time, i.e. $u_3(t) = 1$, $t = 0, 1, \dots$

If the initial sawing capacity of the extra labor is 0, i.e. $x_3(0) = 0$, what is the sawing capacity after three weeks, i.e. $x_3(3)$?

What is the stationary sawing capacity of the extra labor?

Mini problem - solution

Sawing capacity at time $t = 3$:

$$\begin{aligned} x_3(3) &= 0.7x_3(2) + 30u_3(2) = 0.7(0.7x_3(1) + 30u_3(1)) + 30u_3(2) \\ &= 0.7(0.7(0.7x_3(0) + 30u_3(0)) + 30u_3(1)) + 30u_3(2) \\ &= (0.7^3 + 0.7^2 + 0.7 + 1)30 = 65.7 \end{aligned}$$

Stationary capacity is given by:

$$x_3 = 0.7x_3 + 30$$

which gives

$$x_3 = \frac{30}{1 - 0.7} = \frac{30}{0.3} = 100$$

The total sawing capacity is doubled after learning period

Dynamic Production planning example cont'd

The weekly cost for extra personnel is p_3 and p_4 respectively

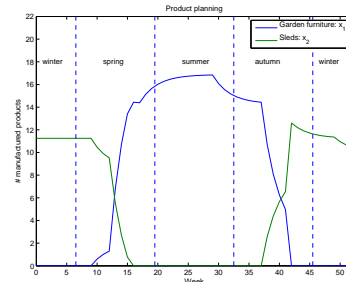
This gives the following production planning problem that optimizes one year ahead production:

$$\begin{aligned} \max \quad & p_1(t)x_1(t) + p_2(t)x_2(t) - p_3(t)u_3(t) - p_4(t)u_4(t) \\ \text{subject to} \quad & x_3(t+1) = 0.7x_3(t) + 30u_3(t) \\ & x_4(t+1) = 0.7x_4(t) + 40.5u_4(t) \\ & 7x_1(t) + 10x_2(t) \leq 100 + x_3(t) \\ & 16x_1(t) + 12x_2(t) \leq 135 + x_4(t) \\ & 0 \leq u_3(t) \leq 1 \quad 0 \leq u_4(t) \leq 1 \\ & x_3(0) = x_3^0 \quad x_4(0) = x_4^0 \end{aligned}$$

for $t = 0, \dots, 52$ and x_3^0 and x_4^0 are the initial capacities for the extra personnel

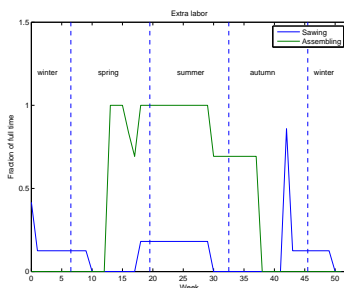
Dynamic Production planning example cont'd

Optimal production over 52 weeks with extra personnel and product prices as before and $p_3 = p_4 = 100$:



Dynamic Production planning example cont'd

Optimal extra labor:



Dynamic Production planning example - limitations

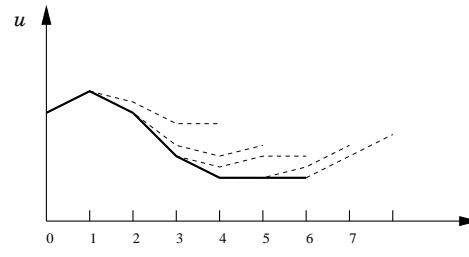
The following is not compensated for:

- Prices may not be equal to predicted prices
- Extra personnel might be fast or slow learners
- Decreased capacity due to employee illness
- ...

Today's lecture

- ▶ Linear Programming (LP)
- ▶ LP in production planning example
- ▶ **Model Predictive Control**
- ▶ A portfolio optimization problem

Model Predictive Control (Receding Horizon Control)



At time t :

1. Measure the state $x(t)$
2. Use model to optimize input trajectory for $t + 1, \dots, t + N$
3. Apply the optimization result $u(t)$ to the system
4. After one sample, go to 1 to repeat the procedure

The History of MPC

- ▶ **A.I. Propoi**, *Use of Linear Programming methods for synthesizing sampled-data automatic systems*, 1963 Automation and Remote Control
- ▶ Used industrially since 1970s, see for example **J. Richalet**, *Model predictive heuristic control — application to industrial processes*, Automatica, 1978.
- ▶ Many industrial products: DMC (Aspen Tech), IDCOM (Adersa), RMPCT (Honeywell), SMCA (Setpoint Inc), SMOC (Shell Global), 3dMPC (ABB), ...
- ▶ Strong theory development since about 1980 (linear) and 1990 (nonlinear)

MPC Example

Product planning example with model-reality mis-match:

Modeled employee learning:

$$\begin{aligned} x_3(t+1) &= 0.7x_3(t) + 30u_3(t) \\ x_4(t+1) &= 0.7x_4(t) + 40.5u_4(t) \end{aligned}$$

Actual employee learning:

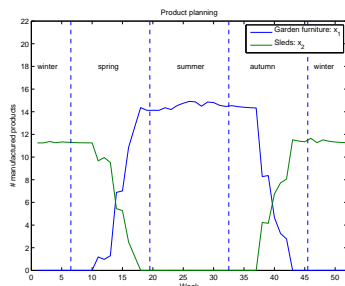
$$\begin{aligned} x_3(t+1) &= 0.75x_3(t) + 30u_3(t) + v_3(t) \\ x_4(t+1) &= 0.65x_4(t) + 40.5u_4(t) + v_4(t) \end{aligned}$$

where $v_3(t)$ and $v_4(t)$ are uniformly distributed random numbers in $[-0.3x_3(t) \ 0]$ and $[-0.3x_4(t) \ 0]$ respectively

The product prices $p_1(t)$ and $p_2(t)$ are additively affected by uniformly distributed random noise in $[-1 \ 1]$

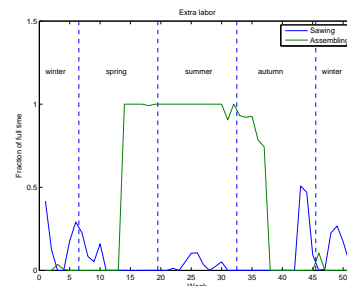
MPC Example - Results

Weekly production when extra labor decided using MPC:



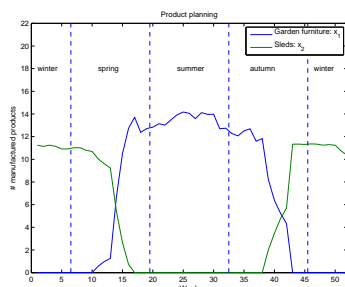
MPC Example - Results

Extra personel (decided using MPC):



MPC Example - Comparison

Production with extra labor as in dynamic production planning example (i.e. no feedback):



Profit over one year is 8.6% higher with MPC-feedback

MPC — Pros and Cons

Pros:

- ▶ Good constraint handling
- ▶ Easily understandable tuning knobs (e.g. cost function)
- ▶ Usually gives good performance in practice
- ▶ Handles complex systems well

Cons:

- ▶ Calculation times
- ▶ System model needed
- ▶ Historically lack of theoretical understanding of the closed loop system

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- ▶ Model Predictive Control
- ▶ **A portfolio optimization problem**

A Dynamic Portfolio of Assets

A portfolio of assets is modelled as

$$\begin{bmatrix} (x_{t+1})_1 \\ \vdots \\ (x_{t+1})_n \end{bmatrix} = \begin{bmatrix} (r_{t+1})_1 & & \\ & \ddots & \\ & & (r_{t+1})_n \end{bmatrix} \begin{bmatrix} (x_t)_1 + (u_t)_1 \\ \vdots \\ (x_t)_n + (u_t)_n \end{bmatrix}$$

or with vector notation $x_{t+1} = R_{t+1}(x_t + u_t)$. Here

- $(x_t)_i$ is the value of asset i at time t
- $(r_{t+1})_i$ is the vector of asset returns, from period t to period $t + 1$
- $(u_t)_i$ is the value of trades in asset i at time t

Assume that r_t for $t = 1, 2, \dots$ are independent random (vector) variables with known mean $\mathbf{E}r_t = \bar{r}_t$ and covariance $\mathbf{E}(r_t - \bar{r}_t)(r_t - \bar{r}_t)^T = \Sigma_t$.

Notation: $\bar{R}_t = \mathbf{E}R_t = \text{diag}(\bar{r}_t)$.

Expressions of interest

- $\mathbf{1}$ a column vector where every entry equals one.
- $\mathbf{1}^T x_t$ the total value of the portfolio before trading at time t
- $\mathbf{1}^T u_t$ the total cash put into the portfolio at time t , *excluding* transaction costs
- $\ell(x_t, u_t)$ the total cost at time t , *including* transaction costs discount factors, etc.
- $-\ell(x_t, u_t)$ the total revenue at time t
- $u_t = \phi_t(x_t)$ The trading policy ϕ_t determines the trades u_t from the portfolio positions x_t

A Portfolio Optimization Problem

Find a trading policy $u_t = \phi_t(x_t)$ that solves the following optimization problem:

$$\text{Minimize} \quad \mathbf{E} \sum_{t=0}^T \ell(x_t, u_t)$$

$$\text{subject to} \quad \begin{cases} x_{t+1} = R_{t+1}(x_t + u_t) \\ u_t = \phi_t(x_t) \end{cases} \quad \text{for } t = 0, 1, \dots, T-1$$

A Portfolio Optimization Problem

In other words, we seek the trading policy ϕ_t that maximizes the total expected revenue.

$$\text{Maximize} \quad -\mathbf{E} \sum_{t=0}^T \ell(x_t, u_t)$$

$$\text{subject to} \quad \begin{cases} x_{t+1} = R_{t+1}(x_t + u_t) \\ u_t = \phi_t(x_t) \end{cases} \quad \text{for } t = 0, 1, \dots, T-1$$

Mini-problem

What would Model Predictive Control mean for the portfolio optimization problem?

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- ▶ More general convex optimization
- ▶ Portfolio optimization revisited
- ▶ Duality and distributed optimization