

# Market Driven Systems (FRTN20)

## Exercise 8 - Solutions

### Game Theory

Last updated: 2014

- Let  $x$  be the number of cars going from start to  $A$  and  $y$  the number of cars going directly from  $A$  to end. Consequently,  $x - y$  cars use the new road. Total travel time is

$$J(x, y) = \frac{x^2}{100} + 45(4000 - x) + 45y + \frac{(4000 - y)^2}{100}.$$

Minimizing with respect to  $x$  and  $y$  gives  $x^* = 2250$  and  $y^* = 1750$ . The new travel time per car is hence

$$\frac{1}{4000}J(x^*, y^*) \approx 64.7,$$

i.e. a gain by 18 s, compared with the situation before the new road and by 15.3 minutes compared to the situation with the road, but without centralized planning.

Comment: Improving a road, or building a new road, can never increase total travel time if centralized planning is used. With decentralized planning it surprisingly can.

- If we use the fact that

$$A_{xy} \geq \min_x A_{xy}, \forall x, y$$

and maximize both sides with respect to  $y$  we get

$$\max_y A_{xy} \geq \max_y \min_x A_{xy}, \forall x.$$

Since this holds for all  $x$ , it also holds for the  $x$  minimizing the left hand side, so

$$\min_x \max_y A_{xy} \geq \max_y \min_x A_{xy}.$$

- Let  $p := P(X = \text{black})$ . The game from  $Y$ 's perspective, knowing  $X$ 's strategy becomes

Y	
red	black
-9(1-p)+5p = 14p-9	5(1-p)-p = -6p+5

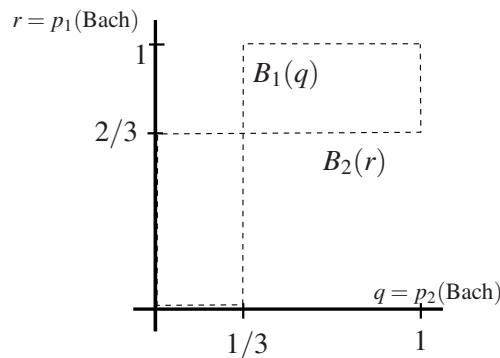
This means that  $Y$  can choose between the two lines in the figure, and will choose the upper one. The optimal choice for the minimizer  $X$  is hence to use the value  $p^* = 0.7$ , since this minimizes the value of the upper line. This will give an outcome of 0.8, and the game hence favors the column player "I".

Since we know  $\min \max = \max \min$  the outcome will be the same analysed from  $X$ 's perspective, knowing  $Y$  strategy. But to verify this let  $q = P(Y = \text{black})$ . This reduces the game to

X	red	-9(1-q)+5q = 14q-9
	black	5(1-q)-q = -6q+5

with optimal solution at  $14q^* - 9 = -6q^* + 5 \Rightarrow q^* = 0.7$ , also giving outcome 0.8.

4. The maximization over  $y$  is equivalent to selecting the maximal element in the vector  $x^T A$ . The second row  $x^T A \leq \alpha \mathbf{1}^T$  upper bounds this maximal element by  $\alpha$ . The bound  $\alpha$  is minimized over  $x$  (first row). The last two lines constrain  $x$  to be a valid probability vector (positive elements, which sum up to 1).
5. Denote the strategies  $r := p_1(\text{Bach})$  and  $q = p_2(\text{Bach})$ .  
 For  $p_2(\text{Bach}) < 1/3$ , player 1 prefers Stravinsky:  $r = 0$ .  
 For  $p_2(\text{Bach}) = 1/3$ , player 1 is indifferent:  $r = [0, 1]$ .  
 For  $p_2(\text{Bach}) > 1/3$ , player 1 prefers Bach:  $r = 1$   
 The best response functions  $r^* = B_1(q)$  and  $q^* = B_2(r)$  are shown in the figure. The intersection(s) are the Nash equilibrium.



6.

- a. Bach and Stravinsky (cf. lecture notes).

		Player 2	
		Bach	Stravinsky
Player 1	Bach	2,1	0,0
	Stravinsky	0,0	1,2

- b. Rock-paper-scissor (cf. the lecture notes).

		Player 2		
		rock	paper	scissor
Player 1	rock	0	1	-1
	paper	-1	0	1
	scissor	1	-1	0

- c. Consider the game below.

		Player 2	
		l	r
Player 1	u	0,0	3,-1
	d	0,0	1,1

Assume first that 1 starts. Choosing  $u$  will result in  $(0,0)$ , while choosing  $d$  will result in  $(1,1)$  (given that 2 is rational). Hence, if 1 starts, the outcome will be  $(1,1)$ . If, instead, 2 starts, the optimal strategies will be  $(l,u)$  or  $(l,d)$ , both resulting in the outcome  $(0,0)$ .

Consequently, both 1 and 2 benefit from 1 being the leader.

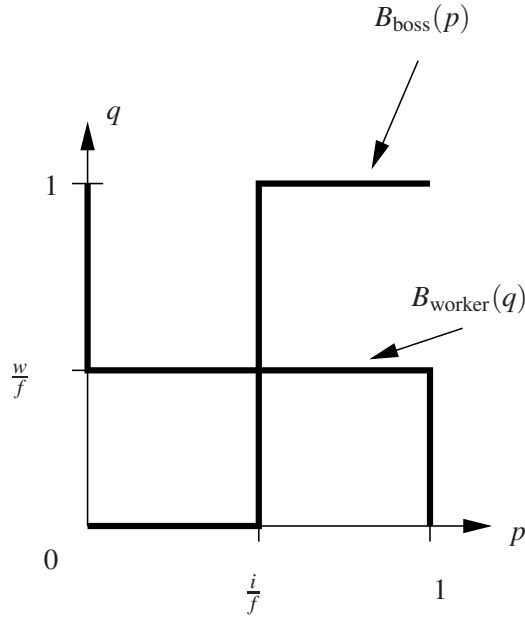
7. Denote by  $p = P(\text{worker shirking})$  and  $q = P(\text{boss inspecting})$ . For the worker, the game then reduces to

worker	work	$-w$
	shirk	$-fq$

Similarly, for the boss the game reduces to

boss	
not inspect	inspect
$g(1-p)$	$g(1-p)+fp-i$

The corresponding best response curves are shown in the figure below.



The Nash equilibrium occurs in the intersection

$$(p^*, q^*) = (\frac{i}{f}, \frac{w}{f}).$$

For the game instance  $w = 1, f = 5, i = 3, g = 1$  this corresponds to the worker working  $3/5$  of the time and the boss inspecting  $1/5$  of the time. The expected outcomes are then

$$U_{\text{worker}} = -w = -1$$

$$U_{\text{boss}} = g(1 - p^*) = \frac{2}{5}.$$

8.

- a. The expected outcome for the boss is now always decreased by inspecting, in fact

$$U_{\text{boss}}(p, q) = g(1-p)(1-q) + (g-i)(1-p)q - ipq = -iq - gp + g$$

which is maximized by  $q^* = 0$ . The situation for the worker is unchanged, and from the best response curve we see that his best response will be  $p^* = 1$ , i.e. shirking.

- b. The boss declares his strategy, defined by  $q$ , whereupon the worker chooses  $p$ . The Stackelberg strategy of the boss is then

$$q^* = \arg \max_q U_{\text{boss}}(q, B_{\text{worker}}(q)) = \arg \max_q (g - gB_{\text{worker}}(q) - iq).$$

From the figure we can see that the best response from the worker is

$$p^* = B_{\text{worker}}(q) = \begin{cases} 1 & , q < \frac{w}{f} \\ [0,1] & , q = \frac{w}{f} \\ 0 & , q > \frac{w}{f} \end{cases}$$

The best response function for  $q = w/f$  is the entire interval, which complicates things somewhat. However by using any  $q > w/f$  one will achieve the response  $p^* = 0$ . This means that the any inspection rate above  $w/f$  will induce work being the optimal response.

The outcome for the boss will then be  $g - iq$  which can be made arbitrarily close to

$$g - iq^* = g - iw/f > 0$$

Due to the fact that the boss needs to declare his inspection rate in advance, an equilibrium has arisen in (work, inspect  $w/f$ ). Note that this situation is more advantageous for the boss than the outcome 0.

9.

- a. The probability density function of each random variable  $X_i \in X = \{X_1, \dots, X_n\}$  is  $f_{X_i}(x) = 1$  and their distribution functions are given by

$$F_{X_i}(x) = \text{Prob}(X_i \leq x) = \int_0^x f_{X_i}(t) dt = x.$$

Let  $Y_k$  be the  $k^{\text{th}}$  largest element in  $X$ . I.e.,  $Y_1$  is the largest, and so on. The distribution function for  $Y_1$  is easily calculated using the observation that  $Y_1 < x$  means that all  $n$  variables are below  $x$ . The probability for this is

$$F_{Y_1}(x) = \prod_{i=1}^n F_{X_i}(x) = x^n,$$

and the density function for the  $Y_1$  can then be calculated from  $f_{Y_1} = \frac{dF_{Y_1}}{dx} = nx^{n-1}$ .

The distribution function for  $Y_k$  is

$$F_{Y_k}(x) = \text{Prob}(Y_k \leq x) = P(\text{at most } k-1 \text{ values are above } x)$$

Splitting the event “at most  $k$  values above  $x$ ” into the disjoint events “at most  $k-1$  values above  $x$ ” and “exactly  $k$  values above  $x$ ” we see that

$$F_{Y_{k+1}}(x) = F_{Y_k}(x) + \binom{n}{k} x^{n-k} (1-x)^k.$$

This recursion gives all distribution functions and density functions, for example

$$f_{Y_2} = \frac{dF_{Y_2}}{dx} = \frac{d}{dx}(x^n + nx^{n-1}(1-x)) = n(n-1)x^{n-2}(1-x)$$

b. We seek

$$m_k = E(Y_K) = \int_0^1 x f_{Y_K}(x) dx = \int_0^1 x \frac{d}{dx} F_{Y_K}(x) dx$$

where the last equality is a consequence of the fundamental theorem of calculus. Combining this with the recursive expression for  $Y_{k+1}$  yields

$$m_{k+1} = m_k + \binom{n}{k} \int_0^1 x \frac{d}{dx} (x^{n-k}(1-x)^k) dx$$

where (by partial integration)

$$\int_0^1 x \frac{d}{dx} (x^{n-k}(1-x)^k) dx = -\frac{k!(n-k)!}{(n+1)!}.$$

Hence

$$m_{k+1} = m_k - \frac{n!}{k!(n-k)!} \cdot \frac{k!(n-k)!}{(n+1)!} = m_k - \frac{1}{n+1}.$$

It is straight forward to compute

$$m_1 = \int_0^1 x \frac{d}{dx} x^n dx = n \int_0^1 x^n dx = \frac{n}{n+1} [x^{n+1}]_0^1 = \frac{n}{n+1}$$

which yields the general result

$$m_k = 1 - \frac{k}{n+1}.$$

10. To confirm that it is a Nash equilibrium, we assume everyone is bidding  $(n-1)/n$  times their valuation and show that it is not beneficial for a single bidder to deviate from this strategy.

The revenue of a bidder with valuation  $V$  and bid  $X$  is

$$\begin{aligned} U(X) &= (V - X) \cdot P(X > \text{all other bids}) = (V - X) \cdot P\left(\frac{n}{n-1}X > \text{all other valuations}\right) \\ &= (V - X) \left(\frac{n}{n-1}X\right)^{n-1}, \quad \text{if } X < (n-1)/n, \quad \text{else } (V - X) \end{aligned}$$

The function is maximized when

$$0 = \frac{dU(X^*)}{dX} \Rightarrow X^* = \frac{n-1}{n}V$$

which shows that the the claim in the problem text is true.

From the definition it is straight forward to find the probability of winning the auction

$$P_i(V_i) = V_i^{n-1}.$$

A result from the lecture gives the expected payoff  $S_i$  through

$$\frac{dS_i}{dV_i} = V_i^n \Rightarrow S_i(V_i) = \frac{V_i^n}{n}.$$