

Lecture 3

- ▶ Phase-plane analysis
- ▶ Classification of singularities
- ▶ Stability of periodic solutions

Material

- ▶ Glad and Ljung: Chapter 13
- ▶ Khalil: Chapter 2.1–2.3
- ▶ Lecture notes

Today's Goal

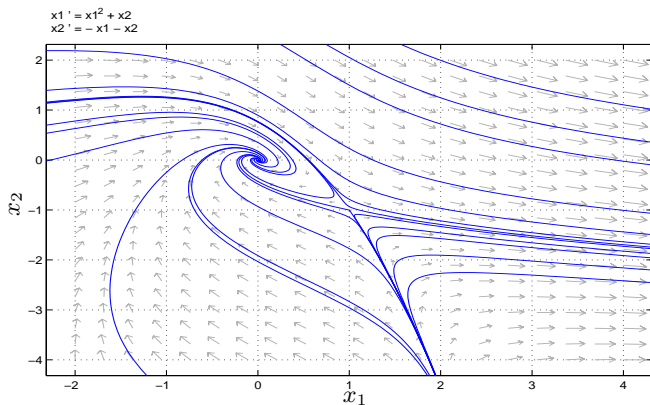
You should be able to

- ▶ *sketch phase portraits for two-dimensional systems*
- ▶ *classify equilibria into nodes, focus, saddle points, and center points.*
- ▶ *analyze limit cycles through Poincaré maps*

First glimpse of phase plane portraits: Consider the system

$$\dot{x}_1 = x_1^2 + x_2$$

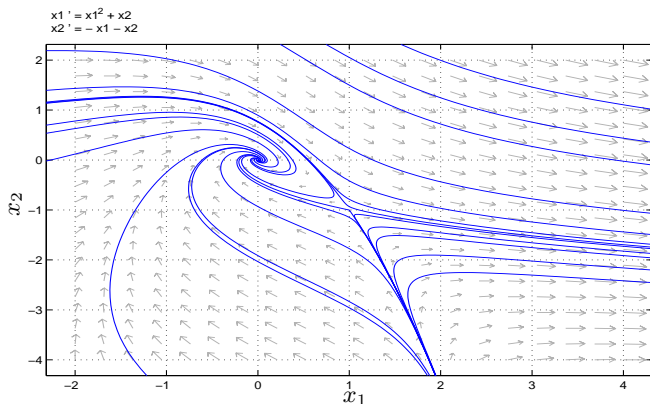
$$\dot{x}_2 = -x_1 - x_2$$



First glimpse of phase plane portraits: Consider the system

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$$\dot{x}_2 = -x_1 - x_2$$

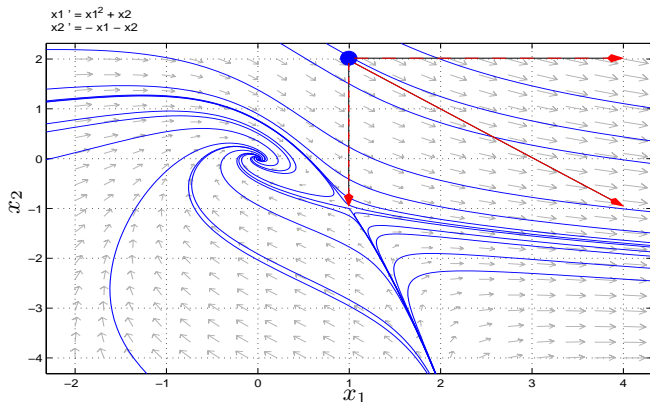


Flow-interpretation: To each point (x_1, x_2) in the plane there is an associated flow-direction $\frac{dx}{dt} = f(x_1, x_2)$

First glimpse of phase plane portraits: Consider the system

$$\dot{x}_1 = x_1^2 + x_2$$

$$\dot{x}_2 = -x_1 - x_2$$



In the point $(x_1, x_2) = (1, 2)$ the **vector field** is pointing in the direction $(1^2 + 2, -1 - 2) = (3, -3)$.

Linear Systems Revival

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analytic solution: $x(t) = e^{At}x(0)$.

If A is diagonalizable, then

$$e^{At} = V e^{\Lambda t} V^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where v_1, v_2 are the eigenvectors of A ($Av_1 = \lambda_1 v_1$ etc).

Matlab:

```
>> [V,Lambda]=eig(A)
```

Example: Two real negative eigenvalues

Given the eigenvalues $\underbrace{\lambda_1}_{\text{faster}} < \underbrace{\lambda_2}_{\text{slower}} < 0$, with corresponding eigenvectors v_1 and v_2 , respectively.

Solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Fast eigenvalue/vector: $x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$ for small t .
Moves along the fast eigenvector for small t

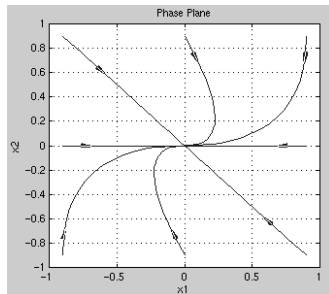
Slow eigenvalue/vector: $x(t) \approx c_2 e^{\lambda_2 t} v_2$ for large t .
Moves along the slow eigenvector towards $x = 0$ for large t

Example—Stable Node

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$$(\lambda_1, \lambda_2) = (-1, -2) \quad \text{and} \quad [v_1 \ v_2] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

v_1 is the slow direction and v_2 is the fast.



Equilibrium Points for Linear Systems

$\text{Im}\lambda_i = 0$: stable node
 $\lambda_1, \lambda_2 < 0$

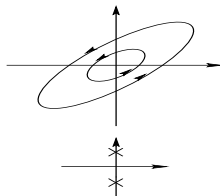
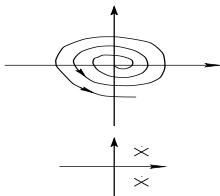
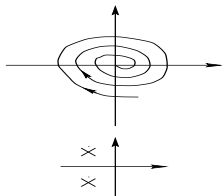
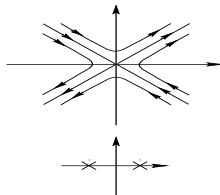
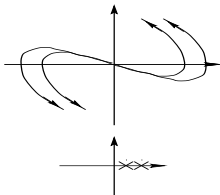
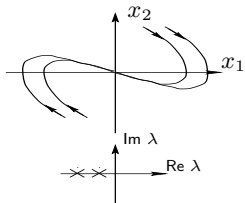
unstable node
 $\lambda_1, \lambda_2 > 0$

saddle point
 $\lambda_1 < 0 < \lambda_2$

$\text{Im}\lambda_i \neq 0$: $\text{Re}\lambda_i < 0$
 stable focus

$\text{Re}\lambda_i > 0$
 unstable focus

$\text{Re}\lambda_i = 0$
 center point



Example—Unstable Focus

$$\dot{x} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x, \quad \sigma, \omega > 0, \quad \lambda_{1,2} = \sigma \pm i\omega$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\sigma t}e^{i\omega t} & 0 \\ 0 & e^{\sigma t}e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} x(0)$$

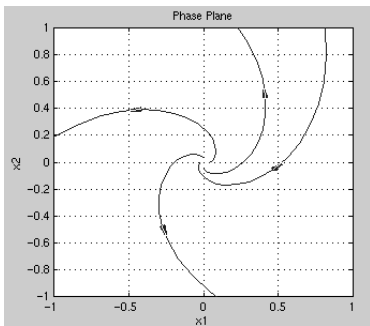
In polar coordinates $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan x_2/x_1$
($x_1 = r \cos \theta$, $x_2 = r \sin \theta$):

$$\dot{r} = \sigma r$$

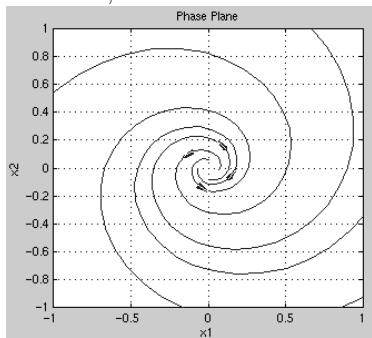
$$\dot{\theta} = \omega$$

Example- unstable focus cont'd

$$\lambda_{1,2} = 1 \pm i$$



$$\lambda_{1,2} = 0.3 \pm i$$



Equilibrium Points for Linear Systems

$\text{Im}\lambda_i = 0$: stable node
 $\lambda_1, \lambda_2 < 0$

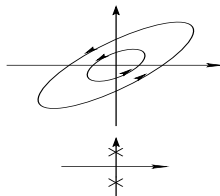
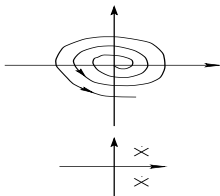
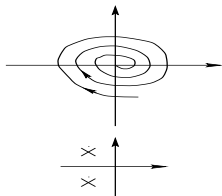
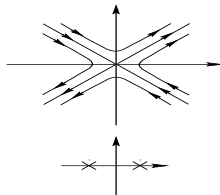
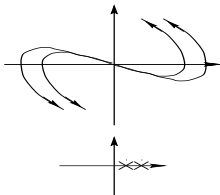
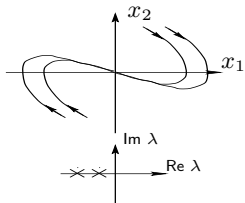
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saddle point
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 stable focus

$\text{Re}\lambda_i > 0$
 unstable focus

$\text{Re}\lambda_i = 0$
 center point



4 minute exercise

What is the phase portrait if $\lambda_1 = \lambda_2$?

Hint: For $\lambda_1 = \lambda_2 = \lambda$ there are two different cases: only one linearly independent eigenvector or all vectors are eigenvectors

Star Node or Multi-Tangent Node

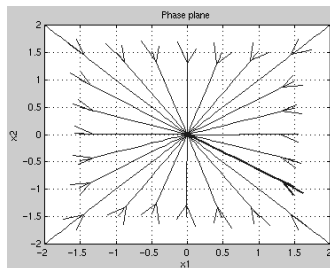
Case I: If

$$\dot{x} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} x, \quad \text{rank}(\lambda I - A) = 0$$

then the solution is

$$x_1(t) = x_1(0)e^{\lambda t}$$

$$x_2(t) = x_2(0)e^{\lambda t}$$



One Tangent Node

Case II: If

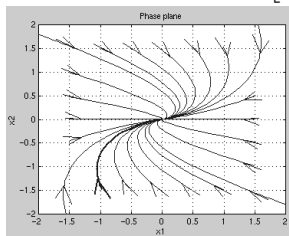
$$\dot{x} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} x, \quad \text{rank}(\lambda I - A) = 1$$

then the solution is

$$x_1(t) = x_1(0)e^{\lambda t} + tx_2(0)e^{\lambda t}$$

$$x_2(t) = x_2(0)e^{\lambda t}$$

There is only one eigenvector: $v_1 = \alpha v_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.



Linear *Time-Varying* Systems (warning)

Warning: Pointwise “Left Half-Plane eigenvalues” of $A(t)$ (*i.e., time-varying systems*) do *NOT* impose stability!!!

$$A(t) = \begin{pmatrix} -1 + \alpha \cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Pointwise eigenvalues are given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

which are in the LHP for $0 < \alpha < 2$ (and here even constant). However,

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t} \cos t & e^{-t} \sin t \\ -e^{(\alpha-1)t} \sin t & e^{-t} \cos t \end{pmatrix} x(0),$$

which is an unbounded solution for $\alpha > 1$.

Phase-Plane Analysis for Nonlinear Systems

Close to equilibria “nonlinear system” \approx “linear system”.

Theorem Assume

$$\dot{x} = f(x)$$

is linearized at x_0 so that

$$\dot{x} = Ax + g(x),$$

where $g \in C^1$ and $\frac{g(x)-g(x_0)}{\|x-x_0\|} \rightarrow 0$ as $x \rightarrow x_0$.

If $\dot{z} = Az$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

If the linearized system has a center, then the nonlinear system has either a center or a focus.

How to Draw Phase Portraits

If done by hand then

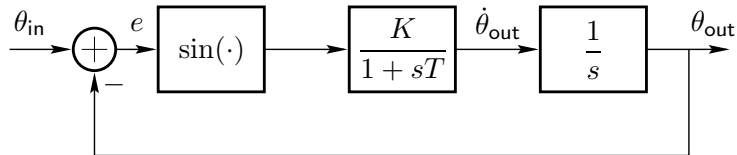
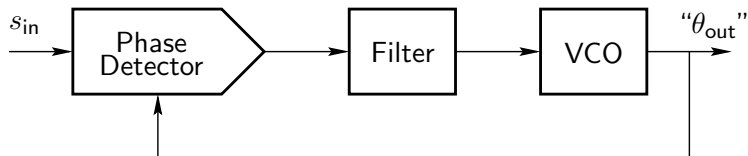
1. Find equilibria (also called singularities)
2. Sketch local behavior around equilibria
3. Sketch (\dot{x}_1, \dot{x}_2) for some other points. Use that $\frac{dx_1}{dx_2} = \frac{\dot{x}_1}{\dot{x}_2}$.
4. Try to find possible limit cycles
5. Guess solutions

Matlab: pptool6/pptool7, dfield6/dfield7, dee, ICTools, etc.

PPTool and some other tools for Matlab is available on or via
<http://www.control.lth.se/course/FRTN05>

Phase-Locked Loop

A PLL tracks phase $\theta_{in}(t)$ of a signal $s_{in}(t) = A \sin[\omega t + \theta_{in}(t)]$.



Singularity Analysis of PLL

Let $x_1(t) = \theta_{\text{out}}(t)$ and $x_2(t) = \dot{\theta}_{\text{out}}(t)$.

Assume $K, T > 0$ and $\theta_{\text{in}}(t) = \theta_{\text{in}}$ constant.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -T^{-1}x_2 + KT^{-1}\sin(\theta_{\text{in}} - x_1)$$

Singularities are $(\theta_{\text{in}} + n\pi, 0)$, since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$

$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{\text{in}} - x_1) = 0 \Rightarrow x_1 = \theta_{\text{in}} + n\pi$$

Singularity Classification of Linearized System

Linearization gives the following characteristic equations:

n even:

$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

$K > (4T)^{-1}$ gives stable focus

$0 < K < (4T)^{-1}$ gives stable node

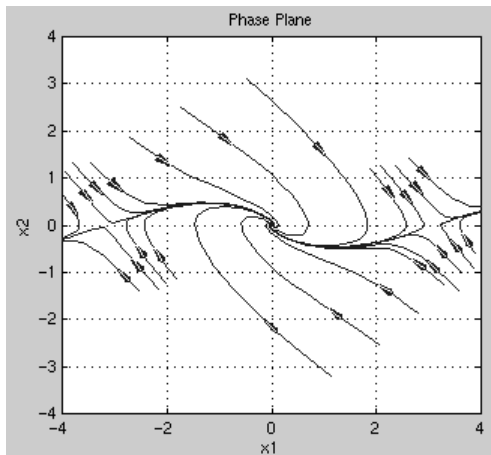
n odd:

$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all $K, T > 0$

Phase-Plane for PLL

$K = 1/2, T = 1$: Focus $(2k\pi, 0)$, saddle points $((2k + 1)\pi, 0)$



Summary

Phase-plane analysis limited to second-order systems (sometimes it is possible for higher-order systems to fix some states)

Many dynamical systems of order three and higher not fully understood (chaotic behaviors etc.)

Bonus — Discrete Time

Many results are parallel (observability, controllability,...)

Example: The difference equation

$$x_{k+1} = f(x_k)$$

is asymptotically stable at x^* if the linearization

$$\left. \frac{\partial f}{\partial x} \right|_{x^*} \text{ has all eigenvalues in } |\lambda| < 1$$

(that is, within the unit circle).

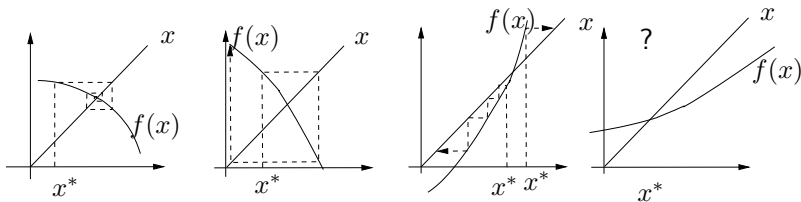
Example (cont'd): Numerical iteration

$$x_{k+1} = f(x_k)$$

to find fixed point

$$x^* = f(x^*)$$

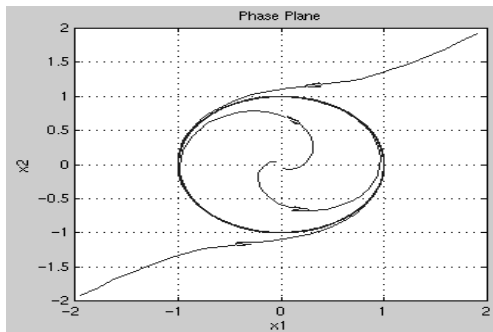
When does the iteration converge?



Periodic Solutions: $x(t + T) = x(t)$

Example of an asymptotically stable periodic solution:

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}\tag{1}$$



Periodic solution: Polar coordinates.

Let

$$x_1 = r \cos \theta \quad \Rightarrow dx_1 = \cos \theta dr - r \sin \theta d\theta$$

$$x_2 = r \sin \theta \quad \Rightarrow dx_2 = \sin \theta dr + r \cos \theta d\theta$$

\Rightarrow

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now

$$\dot{x}_1 = r(1 - r^2) \cos \theta - r \sin \theta$$

$$\dot{x}_2 = r(1 - r^2) \sin \theta + r \cos \theta$$

which gives

$$\dot{r} = r(1 - r^2)$$

$$\dot{\theta} = 1$$

Only $r = 1$ is a stable equilibrium!

A system has a **periodic solution** if for some $T > 0$

$$x(t + T) = x(t), \quad \forall t \geq 0$$

Note that a constant value for $x(t)$ by convention not is regarded as periodic.

- ▶ When does a periodic solution exist?
- ▶ When is it locally (asymptotically) stable? When is it globally asymptotically stable?

Poincaré map (“Stroboscopic map”)

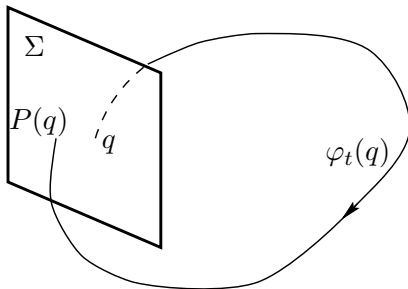
$$\dot{x} = f(x), \quad x \in \mathbf{R}^n$$

$\varphi_t(q)$ is the solution starting in q after time t .

$\Sigma \subset \mathbf{R}^{n-1}$ is a hyperplane transverse to φ_t .

The Poincaré map $P : \Sigma \rightarrow \Sigma$ is

$$P(q) = \varphi_{\tau(q)}(q), \quad \tau(q) \text{ is the first return time}$$

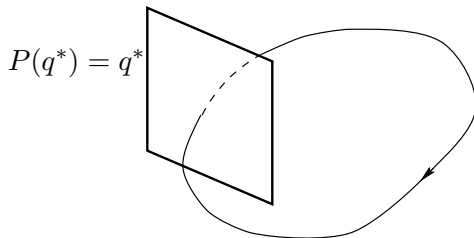


Limit Cycles

If a simple periodic orbit pass through q^* , then $P(q^*) = q^*$.

Such an orbit is called a *limit cycle*.

q^* is called a *fixed point* of P .



Does the iteration $q_{k+1} = P(q_k)$ converge to q^* ?

Locally Stable Limit Cycles

The linearization of P around q^* gives a matrix $W = \left. \frac{\partial P}{\partial q} \right|_{q^*}$ so

$$(q_{k+1} - q^*) \approx W(q_k - q^*),$$

if q_k is close to q^* .

- ▶ If all $|\lambda_i(W)| < 1$, then the corresponding limit cycle is locally **asymptotically stable**.
- ▶ If $|\lambda_i(W)| > 1$, then the limit cycle is **unstable**.

Linearization Around a Periodic Solution

The linearization of

$$\dot{x}(t) = f(x(t))$$

around $x_0(t) = x_0(t + T)$ is

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$$

$$A(t) = \frac{\partial f}{\partial x}(x_0(t)) = A(t + T)$$

P is the map from the solution at $t = 0$ to $t = \tau(q)$.

Example—Stable Unit Circle

Rewrite (1) in polar coordinates:

$$\dot{r} = r(1 - r^2)$$

$$\dot{\theta} = 1$$

Choose $\Sigma = \{(r, \theta) : r > 0, \theta = 2\pi k\}$.

The solution is

$$\varphi_t(r_0, \theta_0) = \left([1 + (r_0^{-2} - 1)e^{-2t}]^{-1/2}, t + \theta_0 \right)$$

First return time from any point $(r_0, \theta_0) \in \Sigma$ is $\tau(r_0, \theta_0) = 2\pi$.

Example—Stable Unit Circle

The Poincaré map is

$$P(r_0) = [1 + (r_0^{-2} - 1)e^{-2 \cdot 2\pi}]^{-1/2}$$

$r_0 = 1$ is a fixed point.

The limit cycle that corresponds to $r(t) = 1$ and $\theta(t) = t$ is locally asymptotically stable, because

$$W = \frac{dP}{dr_0}(1) = [e^{-4\pi}]$$

and

$$|W| = \left| \frac{dP}{dr_0}(1) \right| = |e^{-4\pi}| < 1$$

Example—The Hand Saw

Can we stabilize the inverted pendulum by vertical oscillations?



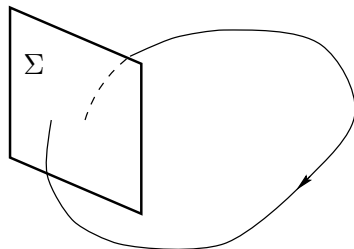
The Hand Saw—Poincaré Map

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{\ell} \left(g + a\omega^2 \sin x_3 \right) \sin x_1$$

$$\dot{x}_3(t) = \omega$$

Choose $\Sigma = \{x_3 = 2\pi k\}$.



The Hand Saw–Poincaré Map

$q^* = 0$ and $T = 2\pi/\omega$. No explicit expression for P . It is, however, easy to determine W numerically. Do two (*or preferably many more*) different simulations with different, small, initial conditions $x(0) = y$ and $x(0) = z$.

Solve W through (*least squares solution of*)

$$\begin{pmatrix} x(T) \Big|_{x(0)=y} & x(T) \Big|_{x(0)=z} \end{pmatrix} = W \begin{pmatrix} y & z \end{pmatrix}$$

This gives for $a = 1\text{cm}$, $\ell = 17\text{cm}$, $\omega = 180$

$$W = \begin{pmatrix} 1.37 & 0.035 \\ -3.86 & 0.630 \end{pmatrix}$$

which has eigenvalues $(1.047, 0.955)$. Unstable.

W is stable for $\omega > 183$

The Hand Saw—Stability Condition

Make the assumptions that

$$\ell \gg a \quad \text{and} \quad a\omega^2 \gg g$$

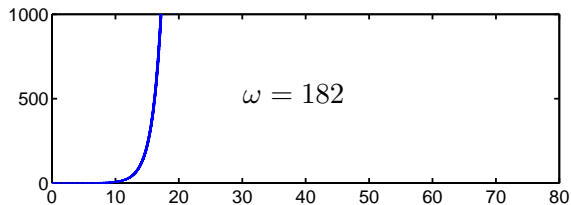
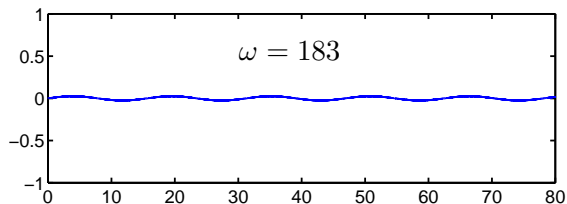
Then some calculations show that the Poincaré map is stable at $q^* = 0$ when

$$\omega > \frac{\sqrt{2g\ell}}{a}$$

$a = 1$ cm and $\ell = 17$ cm give $\omega > 182.6$ rad/s (29 Hz).

The Hand Saw—Simulation

Simulation results give good agreement

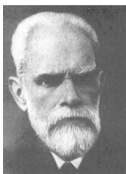


Next Lecture

- ▶ Lyapunov methods for stability analysis

Lyapunov generalized the idea of: *If the total energy is dissipated along the trajectories (i.e the solution curves), the system must be stable.*

Benefit: Might conclude that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.



Lab 1: sign-up starts on Monday

