Lecture 3

- Phase-plane analysis
- Classification of singularities
- Stability of periodic solutions

Material

Glad and Ljung: Chapter 13

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- ▶ Khalil: Chapter 2.1–2.3
- Lecture notes

Today's Goal

You should be able to

- sketch phase portraits for two-dimensional systems
- classify equilibria into nodes, focus, saddle points, and center points.

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analyze limit cycles through Poincaré maps

First glipse of phase plane portraits: Consider the system

$$\dot{x}_1 = x_1^2 + x_2 \dot{x}_2 = -x_1 - x_2$$



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First glipse of phase plane portraits: Consider the system

$$\dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = -x_1 - x_2$$



Flow-interpretation: To each point (x_1, x_2) in the plane there is an associated flow-direction $\frac{dx}{dt} = f(x_1, x_2)$

First glipse of phase plane portraits: Consider the system

$$\dot{x}_1 = x_1^2 + x_2 \dot{x}_2 = -x_1 - x_2$$



In the point $(x_1, x_2) = (1, 2)$ the vector field is pointing in the direction $(1^2 + 2, -1 - 2) = (3, -3)$.

Vectorfields in Oceanography ...



On 10 January 1992, during a storm in the North Pacific Ocean close to the International Date Line, twelve 40-foot (13.3 m) intermodal containers were washed overboard. One of these containers held 28,800 Floatees,...



http://en.wikipedia.org/wiki/Friendly_Floatees

Linear Systems Revival

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analytic solution: $x(t) = e^{At}x(0)$. If A is diagonalizable, then

$$e^{At} = V e^{\Lambda t} V^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where v_1, v_2 are the eigenvectors of A $(Av_1 = \lambda_1 v_1 \text{ etc})$.

Matlab:

>> [V,Lambda]=eig(A)

Example: Two real negative eigenvalues

Given the eigenvalues $\lambda_1 < \lambda_2 < 0$, with corresponding eigenvectors v_1 and v_2 , respectively.

Solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Fast eigenvalue/vector: $x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$ for small t. Moves along the fast eigenvector for small t

Slow eigenvalue/vector: $x(t) \approx c_2 e^{\lambda_2 t} v_2$ for large t. Moves along the slow eigenvector towards x = 0 for large t

Example—Stable Node

$$\begin{split} \dot{x} &= \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x \\ (\lambda_1, \lambda_2) &= (-1, -2) \quad \text{and} \quad \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \end{split}$$

 v_1 is the slow direction and v_2 is the fast.



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Equilibrium Points for Linear Systems



Example—Unstable Focus

$$\begin{split} \dot{x} &= \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x, \qquad \sigma, \omega > 0, \qquad \lambda_{1,2} = \sigma \pm i\omega \\ x(t) &= e^{At}x(0) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\sigma t}e^{i\omega t} & 0 \\ 0 & e^{\sigma t}e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1}x(0) \\ \\ & \text{In polar coordinates } r = \sqrt{x_1^2 + x_2^2}, \ \theta = \arctan x_2/x_1 \\ (x_1 = r\cos\theta, \ x_2 = r\sin\theta): \\ & \cdot \end{split}$$

 $\dot{r} = \sigma r$ $\dot{\theta} = \omega$

Example- unstable focus cont'd

 $\lambda_{1,2} = 1 \pm i$





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Equilibrium Points for Linear Systems



What is the phase portrait if $\lambda_1 = \lambda_2$?

Hint: For $\lambda_1 = \lambda_2 = \lambda$ there are two different cases: only one linearly independent eigenvector or all vectors are eigenvectors

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Star Node or Multi-Tangent Node

Case I: If
$$\dot{x} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} x, \qquad {\rm rank}\,(\lambda I - A) = 0$$

then the solution is

$$x_1(t) = x_1(0)e^{\lambda t}$$
$$x_2(t) = x_2(0)e^{\lambda t}$$



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One Tangent Node

Case II: If
$$\dot{x} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} x, \qquad \mathrm{rank}\,(\lambda I - A) = 1$$

then the solution is

$$x_1(t) = x_1(0)e^{\lambda t} + tx_2(0)e^{\lambda t}$$
$$x_2(t) = x_2(0)e^{\lambda t}$$

There is only one eigenvector: $v_1 = \alpha v_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.



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Linear *Time-Varying* Systems (warning)

Warning: Pointwise "Left Half-Plane eigenvalues" of A(t) (*i.e.*, *time-varying systems*) do *NOT* impose stability!!!

$$A(t) = \begin{pmatrix} -1 + \alpha \cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Pointwise eigenvalues are given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

which are in the LHP for $0 < \alpha < 2$ (and here even constant). However,

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{pmatrix} x(0),$$

which is an unbounded solution for $\alpha > 1$.

Phase-Plane Analysis for Nonlinear Systems

Close to equilibria "nonlinear system" \approx "linear system". Theorem Assume

$$\dot{x} = f(x)$$

is linearized at x_0 so that

$$\dot{x} = Ax + g(x),$$

where $g \in C^1$ and $\frac{g(x)-g(x_0)}{\|x-x_0\|} \to 0$ as $x \to x_0$. If $\dot{z} = Az$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin. If the linearized system has a center, then the nonlinear system has either a center or a focus.

How to Draw Phase Portraits

If done by hand then

- 1. Find equilibria (also called singularities)
- 2. Sketch local behavior around equilibria
- 3. Sketch (\dot{x}_1, \dot{x}_2) for some other points. Use that $\frac{dx_1}{dx_2} = \frac{\dot{x}_1}{\dot{x}_2}$.
- 4. Try to find possible limit cycles
- 5. Guess solutions

Phase-Locked Loop

A PLL tracks phase $\theta_{in}(t)$ of a signal $s_{in}(t) = A \sin[\omega t + \theta_{in}(t)]$.



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Singularity Analysis of PLL

Let $x_1(t) = \theta_{out}(t)$ and $x_2(t) = \dot{\theta}_{out}(t)$. Assume K, T > 0 and $\theta_{in}(t) = \theta_{in}$ constant.

$$\dot{x}_1 = x_2$$

 $\dot{x}_2 = -T^{-1}x_2 + KT^{-1}\sin(\theta_{in} - x_1)$

Singularities are $(\theta_{\rm in}+n\pi,0),$ since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$

$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{in} - x_1) = 0 \Rightarrow x_1 = \theta_{in} + n\pi$$

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Singularity Classification of Linearized System

Linearization gives the following characteristic equations: $\underline{n \text{ even}}$:

$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

 $K > (4T)^{-1}$ gives stable focus $0 < K < (4T)^{-1}$ gives stable node

<u>n odd:</u>

$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all K, T > 0

Phase-Plane for PLL

K = 1/2, T = 1: Focus $(2k\pi, 0)$, saddle points $((2k+1)\pi, 0)$



Summary

Phase-plane analysis limited to second-order systems (sometimes it is possible for higher-order systems to fix some states) Many dynamical systems of order three and higher not fully understood (chaotic behaviors etc.)

Bonus — Discrete Time

Many results are parallel (observability, controllability,...)

Example: The difference equation

$$x_{k+1} = f(x_k)$$

is asymptotically stable at \boldsymbol{x}^* if the linearization

$$\left. rac{\partial f}{\partial x}
ight|_{x^*}$$
 has all eigenvalues in $|\lambda| < 1$

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(that is, within the unit circle).

Example (cont'd): Numerical iteration

$$x_{k+1} = f(x_k)$$

to find fixed point

$$x^* = f(x^*)$$

When does the iteration converge?



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Periodic Solutions: x(t+T) = x(t)

Example of an asymptotically stable periodic solution:

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)
\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2)$$
(1)

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Periodic solution: Polar coordinates.

Let

$$\begin{aligned} x_1 &= r\cos\theta \quad \Rightarrow dx_1 &= \cos\theta dr - r\sin\theta d\theta \\ x_2 &= r\sin\theta \quad \Rightarrow dx_2 &= \sin\theta dr + r\cos\theta d\theta \end{aligned}$$

 \Rightarrow

$$\left(\begin{array}{c} \dot{r} \\ \dot{\theta} \end{array}\right) = \frac{1}{r} \left(\begin{array}{c} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{array}\right) \left(\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array}\right)$$

Now

$$\dot{x}_1 = r(1 - r^2)\cos\theta - r\sin\theta$$
$$\dot{x}_2 = r(1 - r^2)\sin\theta + r\cos\theta$$

which gives

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Only r = 1 is a stable equilibrium!

A system has a **periodic solution** if for some T > 0

$$x(t+T) = x(t), \quad \forall t \ge 0$$

Note that a constant value for x(t) by convention not is regarded as periodic.

- When does a periodic solution exist?
- When is it locally (asymptotically) stable? When is it globally asymptotically stable?

Poincaré map ("Stroboscopic map")

$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n$$

 $\varphi_t(q)$ is the solution starting in q after time t. $\Sigma \subset \mathbf{R}^{n-1}$ is a hyperplane transverse to φ_t . The Poincaré map $P: \Sigma \to \Sigma$ is

 $P(q) = arphi_{ au(q)}(q), \qquad au(q) ext{ is the first return time}$



Limit Cycles

If a simple periodic orbit pass through q^* , then $P(q^*) = q^*$. Such an orbit is called a *limit cycle*. q^* is called a *fixed point* of P.



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Does the iteration $q_{k+1} = P(q_k)$ converge to q^* ?

Locally Stable Limit Cycles

The linearization of P around q^* gives a matrix $W = \frac{\partial P}{\partial q}\Big|_{q^*}$ so

$$(q_{k+1} - q^*) \approx W(q_k - q^*),$$

if q_k is close to q^* .

If all |λ_i(W)| < 1, then the corresponding limit cycle is locally asymptotically stable.</p>

• If $|\lambda_i(W)| > 1$, then the limit cycle is **unstable**.

Linearization Around a Periodic Solution

The linearization of

$$\dot{x}(t) = f(x(t))$$

around $x_0(t) = x_0(t+T)$ is $\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$ $A(t) = \frac{\partial f}{\partial x}(x_0(t)) = A(t+T)$

P is the map from the solution at t = 0 to $t = \tau(q)$.

Example—Stable Unit Circle

Rewrite (1) in polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Choose $\Sigma = \{(r, \theta) : r > 0, \theta = 2\pi k\}.$ The solution is

$$\varphi_t(r_0, \theta_0) = \left([1 + (r_0^{-2} - 1)e^{-2t}]^{-1/2}, t + \theta_0 \right)$$

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First return time from any point $(r_0, \theta_0) \in \Sigma$ is $\tau(r_0, \theta_0) = 2\pi$.

Example—Stable Unit Circle

The Poincaré map is

$$P(r_0) = [1 + (r_0^{-2} - 1)e^{-2 \cdot 2\pi}]^{-1/2}$$

 $r_0 = 1$ is a fixed point.

The limit cycle that corresponds to r(t)=1 and $\theta(t)=t$ is locally asymptotically stable, because

$$W = \frac{dP}{dr_0}(1) = \left[e^{-4\pi}\right]$$

and

$$|W| = \left|\frac{dP}{dr_0}(1)\right| = |e^{-4\pi}| < 1$$

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Example—The Hand Saw

Can we stabilize the inverted pendulum by vertical oscillations?



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The Hand Saw—Poincaré Map

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{\ell} \left(g + a\omega^2 \sin x_3 \right) \sin x_1$$
$$\dot{x}_3(t) = \omega$$

Choose $\Sigma = \{x_3 = 2\pi k\}.$



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The Hand Saw–Poincaré Map

 $q^* = 0$ and $T = 2\pi/\omega$. No explicit expression for P. It is, however, easy to determine W numerically. Do two (or preferably many more) different simulations with different, small, initial conditions x(0) = y and x(0) = z. Solve W through (least squares solution of)

$$\begin{pmatrix} x(T) \Big|_{x(0)=y} & x(T) \Big|_{x(0)=z} \end{pmatrix} = W \begin{pmatrix} y & z \end{pmatrix}$$

This gives for $a=1 {\rm cm},~\ell=17 {\rm cm},~\omega=180$

$$W = \begin{pmatrix} 1.37 & 0.035 \\ -3.86 & 0.630 \end{pmatrix}$$

which has eigenvalues (1.047, 0.955). Unstable. W is stable for $\omega > 183$

The Hand Saw—Stability Condition

Make the assumptions that

$$\ell \gg a$$
 and $a\omega^2 \gg g$

Then some calculations show that the Poincaré map is stable at $q^* = 0$ when

$$\omega > \frac{\sqrt{2g\ell}}{a}$$

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 $a = 1 \text{ cm and } \ell = 17 \text{ cm give } \omega > 182.6 \text{ rad/s}$ (29 Hz).

The Hand Saw—Simulation

Simulation results give good agreement



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Next Lecture

Lyapunov methods for stability analysis

Lyapunov generalized the idea of: *If the total energy is dissipated along the trajectories (i.e the solution curves), the system must be stable.*

Benefit: Might conclude that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.



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Lab 1: sign-up starts on Monday



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