

# Nonlinear Control and Servo systems

## Lecture 1

Giacomo Como, 2014

Dept. of Automatic Control  
LTH, Lund University

November 2, 2015

# Overview Lecture 1

- Practical information
- Course contents
- Nonlinear control systems phenomena
- Nonlinear differential equations

# Course Goal

*To provide students with solid theoretical foundations of nonlinear control systems combined with good engineering ability*

You should after the course be able to

- ▶ recognize common nonlinear control problems,
- ▶ use some powerful analysis methods, and
- ▶ use some practical design methods.

# Today's Goal

- ▶ *Recognize some common nonlinear phenomena*
- ▶ *Transform differential equations to autonomous form, first-order form, and feedback form*
- ▶ *Describe saturation, dead-zone, relay with hysteresis, backlash*
- ▶ *Calculate equilibrium points*

# Course Material

## ► Textbook

- Glad and Ljung, *Reglerteori, flervariabla och olinjra metoder*, 2003, Studentlitteratur, ISBN 9-14-403003-7 or the English translation *Control Theory*, 2000, Taylor & Francis Ltd, ISBN 0-74-840878-9. The course covers Chapters 11-16,18. (MPC and optimal control not covered in the other alternative textbooks.)
- H. Khalil, *Nonlinear Systems* (3rd ed.), 2002, Prentice Hall, ISBN 0-13-122740-8. A good, a bit more advanced text.
- ALTERNATIVE: Slotine and Li, *Applied Nonlinear Control*, Prentice Hall, 1991. The course covers chapters 1-3 and 5, and sections 4.7-4.8, 6.2, 7.1-7.3.

# Course Material, cont.

- ▶ Handouts (Lecture notes + extra material)
- ▶ Exercises (can be downloaded from the course home page)
- ▶ Lab PMs 1, 2 and 3
- ▶ Home page  
`http://www.control.lth.se/course/FRTN05/`
- ▶ Matlab/Simulink other simulation software  
see home page

# Lectures and labs

The lectures (28 hours) are given as follows:

Mon 13–15,	E:C	Nov 2 – Dec 7
Wed 8–10,	E:C	Nov 4 – Dec 9
<b>Fri 8-10</b>	E:C	<b>Nov 6</b>
<b>Thu 8-10</b>	<b>M:2112B</b>	<b>Dec 10</b>



Lectures are given in English.

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The three laboratory experiments are [mandatory](#).

[Sign-up lists](#) are posted [on the web](#) at least one week before the first laboratory experiment. *The lists close one day before the first session.*

The Laboratory PMs are available at the course homepage.

[Before the lab](#) sessions some [home assignments](#) have to be done.

No reports after the labs.



## Exercise sessions and TAs

The exercises (28 hours) are offered twice a week

Tue 15:15-17:00 **M:2112B**

Wed 15:15-17:00 **M:2112B**

NOTE: The exercises are held in the seminar room of the Automatic Control Department, M-building, second floor **see schedule on home page.**

EXCEPTIONS: (i) for the first two weeks only Wednesday exercise sessions are scheduled at 13:15–15:00 instead of 15:15-17:00.

(ii) on November 24 and 25 the exercise sessions will be held in a different room to be announced in due time

Christian Grussler



Olof Troeng



Mahdi Ghazei



# The Course

- ▶ 14 lectures
- ▶ 14 exercises
- ▶ 3 laboratories
- ▶ 5 hour exam: **January 13, 2015, 14:00-19:00, MA10 I-J.**  
Open-book exam: Lecture notes but no old exams or exercises allowed.

# Course Outline

- Lecture 1-3     Modelling and basic phenomena  
(linearization, phase plane, limit cycles)
- Lecture 2-6     Analysis methods  
(Lyapunov, circle criterion, describing functions))
- Lecture 7-8     Common nonlinearities  
(Saturation, friction, backlash, quantization))
- Lecture 9-13    Design methods  
(Lyapunov methods, Backstepping, Optimal control)
- Lecture 14      Summary

# Today's lecture

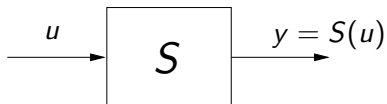
## Common nonlinear phenomena

- ▶ Input-dependent stability
- ▶ Stable periodic solutions
- ▶ Jump resonances and subresonances

## Nonlinear model structures

- ▶ Common nonlinear components
- ▶ State equations
- ▶ Feedback representation

# Linear Systems



**Definitions:** The system  $S$  is *linear* if

$$S(\alpha u) = \alpha S(u), \quad \text{scaling}$$

$$S(u_1 + u_2) = S(u_1) + S(u_2), \quad \text{superposition}$$

A system is *time-invariant* if delaying the input results in a delayed output:

$$y(t - \tau) = S(u(t - \tau))$$

# Linear time-invariant systems are easy to analyze

Different representations of same system/behavior

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0$$

$$y(t) = g(t) \star u(t) = \int g(r)u(t-r)dr$$

$$Y(s) = G(s)U(s)$$

Local stability = global stability:

Eigenvalues of  $A$  (= poles of  $G(s)$ ) in left half plane

Superposition:

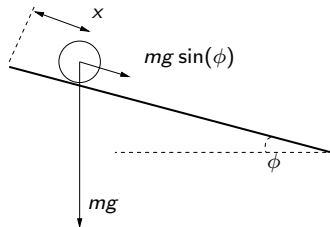
Enough to know step (or impulse) response

Frequency analysis possible:

Sinusoidal inputs give sinusoidal outputs

# Linear models are not always enough

## Example: Ball and beam



Linear model (acceleration along beam) :

Combine  $F = m \cdot a = m \frac{d^2 x}{dt^2}$  with  $F = mg \sin(\phi)$ :

$$\ddot{x}(t) = g \sin(\phi(t))$$

## Linear models are not enough

$x$  = position (m)       $\phi$  = angle (rad)       $g = 9.81 \text{ (m/s}^2\text{)}$

Can the ball move 0.1 meter in 0.1 seconds with constant  $\phi$ ?



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Linearization:  $\sin \phi \sim \phi$  for  $\phi \sim 0$

$$\begin{cases} \ddot{x}(t) = g\phi \\ x(0) = 0 \end{cases}$$

Solving the above gives  $x(t) = \frac{t^2}{2}g\phi$

For  $x(0.1) = 0.1$ , one needs  $\phi = \frac{2*0.1}{0.1^2*g} \geq 2 \text{ rad}$

Clearly outside linear region!

Contact problem, friction, centripetal force, saturation

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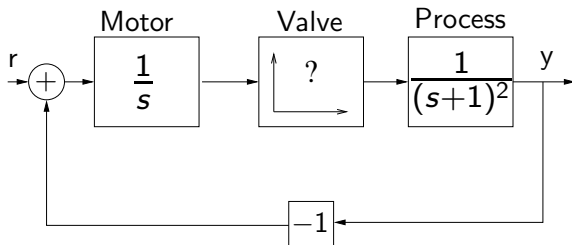
How fast can it be done? (Optimal control)

# Warm-Up Exercise: 1-D Nonlinear Control System

$$\dot{x} = x^2 - x + u$$

- ▶ stability for  $u = 0$ ?
- ▶ stability for constant  $u = b$ ?
- ▶ stability with linear feedback  $u = ax + b$ ?
- ▶ stability with non-linear feedback  $u(x) = ?$

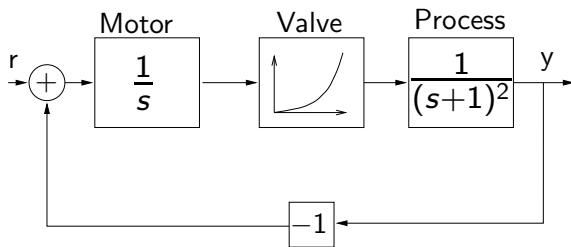
# Stability Can Depend on Amplitude



Valve characteristic  $f(x) = ???$

Step changes of amplitude,  $r = 0.2$ ,  $r = 1.68$ , and  $r = 1.72$

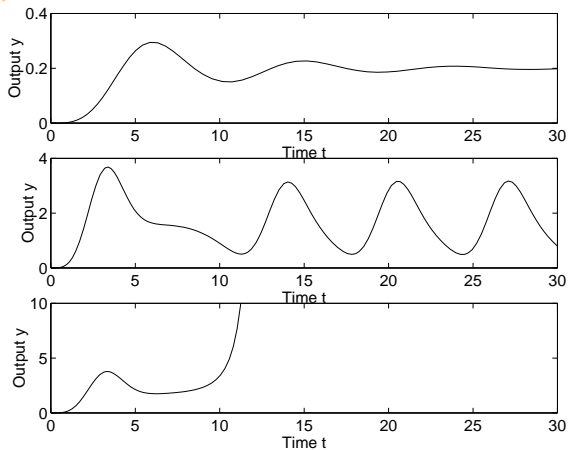
# Stability Can Depend on Amplitude



Valve characteristic  $f(x) = x^2$

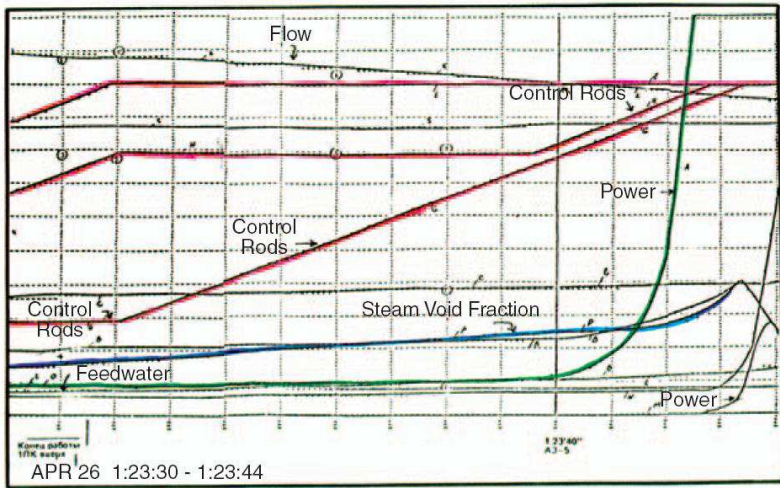
Step changes of amplitude,  $r = 0.2$ ,  $r = 1.68$ , and  $r = 1.72$

# Step Responses



$r = 0.2$

$r = 1.68$



$$\sim e^{2t}$$

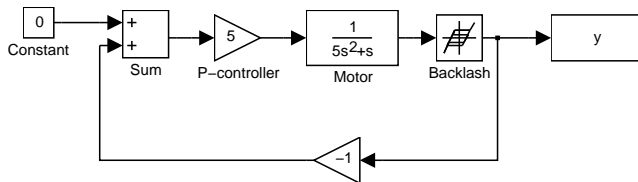


**Figure 2.** *Chernobyl nuclear power plant shortly after the accident on 26 April 1986.*



# Stable Periodic Solutions

## Example: Motor with back-lash

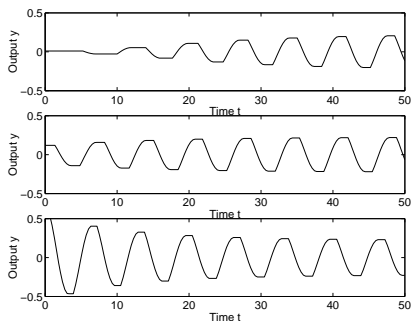


Motor:  $G(s) = \frac{1}{s(1+5s)}$

Controller:  $K = 5$

# Stable Periodic Solutions

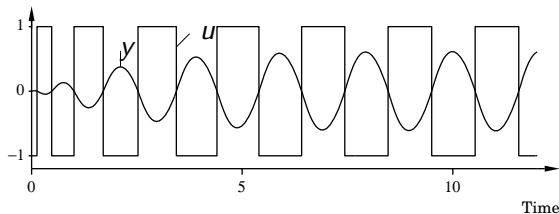
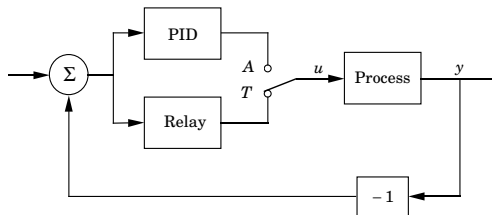
Output for different initial conditions:



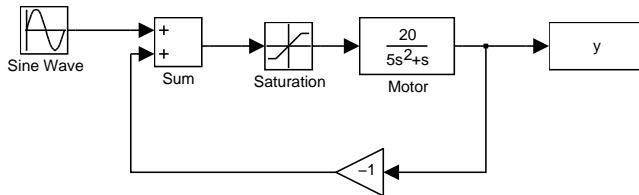
Frequency and amplitude independent of initial conditions!  
Several systems use the existence of such a phenomenon

# Relay Feedback Example

Period and amplitude of limit cycle are used for autotuning



# Jump Resonances

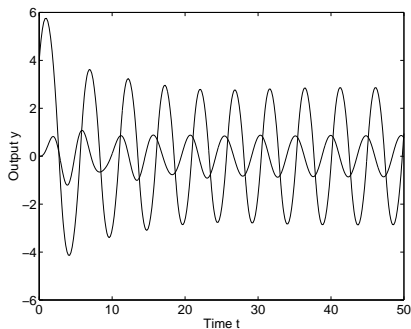


Response for sinusoidal depends on initial condition  
Problem when doing frequency response measurement

# Jump Resonances

$$u = 0.5 \sin(1.3t), \quad \text{saturation level} = 1.0$$

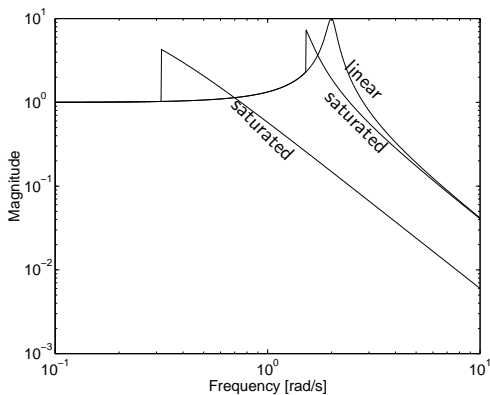
Two different initial conditions



give two different amplifications for same sinusoid!

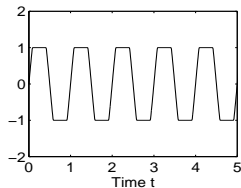
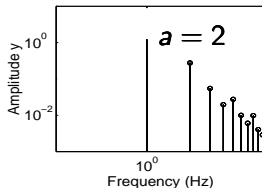
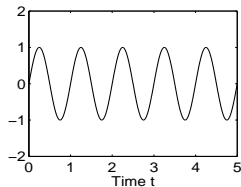
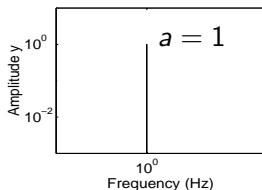
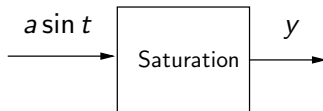
# Jump Resonances

Measured frequency response (many-valued)



# New Frequencies

**Example:** Sinusoidal input, saturation level 1



# New Frequencies

**Example:** Electrical power distribution

$$\text{THD} = \text{Total Harmonic Distortion} = \frac{\sum_{k=2}^{\infty} \text{energy in tone } k}{\text{energy in tone } 1}$$

Nonlinear loads: Rectifiers, switched electronics, transformers

Important, increasing problem

Guarantee electrical quality

Standards, such as  $\text{THD} < 5\%$





# New Frequencies

**Example:** Mobile telephone

Effective amplifiers work in nonlinear region

Introduces spectrum leakage

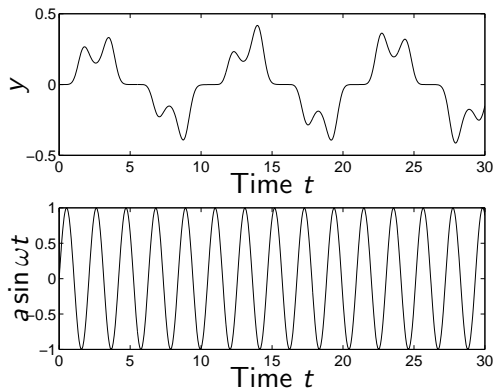
Channels close to each other

Trade-off between effectivity and linearity



# Subresonances

**Example:** Duffing's equation  $\ddot{y} + \dot{y} + y - y^3 = a \sin(\omega t)$



# When is Nonlinear Theory Needed?

- ▶ Hard to know when - Try simple things first!
- ▶ Regulator problem versus servo problem
- ▶ Change of working conditions (production on demand, short batches, many startups)
- ▶ Mode switches
- ▶ Nonlinear components

How to detect? Make step responses, Bode plots

- ▶ Step up/step down
- ▶ Vary amplitude
- ▶ Sweep frequency up/frequency down

# Some Nonlinearities

Static – dynamic



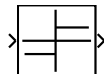
Abs



Math  
Function



Saturation



Sign



Dead Zone



Look-Up  
Table



Relay



Backlash



Coulomb &  
Viscous Friction

# Nonlinear Differential Equations

## Problems

- ▶ No analytic solutions
- ▶ Existence?
- ▶ Uniqueness?
- ▶ etc

# Finite escape time

**Example:** The differential equation

$$\frac{dx}{dt} = x^2, \quad x(0) = x_0$$

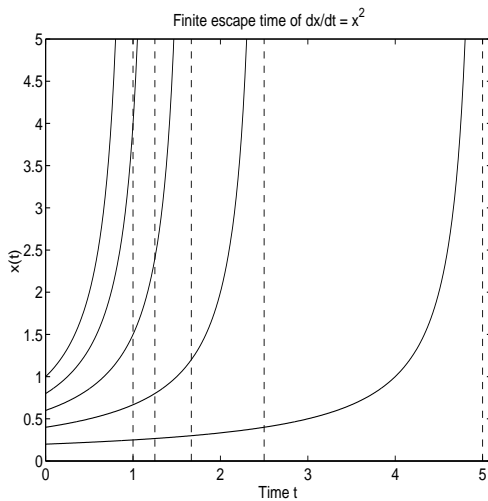
has solution

$$x(t) = \frac{x_0}{1 - x_0 t}, \quad 0 \leq t < \frac{1}{x_0}$$

Finite escape time

$$t_f = \frac{1}{x_0}$$

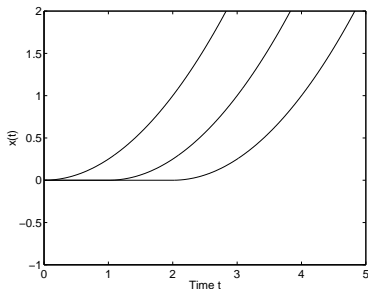
# Finite Escape Time



# Uniqueness Problems

**Example:** The equation  $\dot{x} = \sqrt{x}$ ,  $x(0) = 0$  has many solutions:

$$x(t) = \begin{cases} (t - C)^2/4 & t > C \\ 0 & t \leq C \end{cases}$$



Compare with water tank:

$$dh/dt = -a\sqrt{h}, \quad h : \text{height (water level)}$$

Change to backward-time: “If I see it empty, when was it full?”



# Local Existence and Uniqueness

For  $R > 0$ , let  $\Omega_R$  denote the ball  $\Omega_R = \{z : \|z - a\| \leq R\}$ .

## Theorem

If,  $f$  is Lipschitz-continuous in  $\Omega_R$ , i.e.,

$$\|f(z) - f(y)\| \leq K\|z - y\|, \quad \text{for all } z, y \in \Omega_R,$$

then

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases}$$

has a unique solution

$$x(t), \quad 0 \leq t < R/C_R,$$

where  $C_R = \max_{x \in \Omega_R} \|f(x)\|$

# Global Existence and Uniqueness

## Theorem

If  $f$  is Lipschitz-continuous in  $R^n$ , i.e.,

$$\|f(z) - f(y)\| \leq K\|z - y\|, \quad \text{for all } z, y \in R^n,$$

then

$$\dot{x}(t) = f(x(t)), x(0) = a$$

has a unique solution

$$x(t), \quad t \geq 0.$$

# State-Space Models

- ▶ State vector  $x$
- ▶ Input vector  $u$
- ▶ Output vector  $y$

general:  $f(x, u, y, \dot{x}, \dot{u}, \dot{y}, \dots) = 0$

explicit:  $\dot{x} = f(x, u), \quad y = h(x)$

affine in  $u$ :  $\dot{x} = f(x) + g(x)u, \quad y = h(x)$

linear time-invariant:  $\dot{x} = Ax + Bu, \quad y = Cx$

# Transformation to Autonomous System

Nonautonomous:

$$\dot{x} = f(x, t)$$

Always possible to transform to autonomous system

Introduce  $x_{n+1} = \text{time}$

$$\begin{aligned}\dot{x} &= f(x, x_{n+1}) \\ \dot{x}_{n+1} &= 1\end{aligned}$$

# Transformation to First-Order System

Assume  $\frac{d^k y}{dt^k}$  highest derivative of  $y$

Introduce  $x = \left[ y \quad \frac{dy}{dt} \quad \dots \quad \frac{d^{k-1}y}{dt^{k-1}} \right]^T$

**Example:** Pendulum

$$MR\ddot{\theta} + k\dot{\theta} + MgR \sin \theta = 0$$

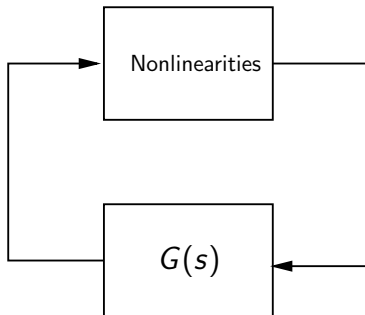
$x = \left[ \theta \quad \dot{\theta} \right]^T$  gives

$$\dot{x}_1 = x_2$$

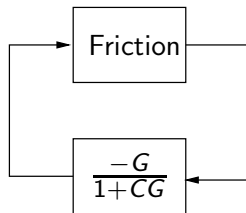
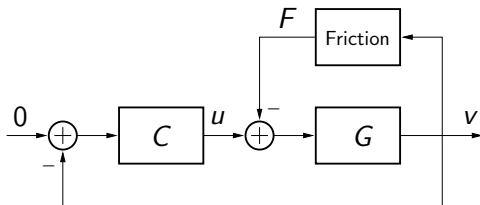
$$\dot{x}_2 = -\frac{k}{MR}x_2 - \frac{g}{R} \sin x_1$$

# A Standard Form for Analysis

Transform to the following form



## Example, Closed Loop with Friction



# Equilibria (=singular points)

*Put all derivatives to zero!*

General:  $f(x_0, u_0, y_0, 0, 0, 0, \dots) = 0$

Explicit:  $f(x_0, u_0) = 0$

Linear:  $Ax_0 + Bu_0 = 0$  (has analytical solution(s)!)



# Multiple Equilibria

Example: Pendulum

$$MR\ddot{\theta} + k\dot{\theta} + MgR \sin \theta = 0$$

Equilibria given by  $\ddot{\theta} = \dot{\theta} = 0 \implies \sin \theta = 0 \implies \theta = n\pi$

Alternatively,

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{MR}x_2 - \frac{g}{R}\sin x_1\end{aligned}$$

gives  $x_2 = 0$ ,  $\sin(x_1) = 0$ , etc

# Next Lecture

- ▶ Linearization
- ▶ Stability definitions
- ▶ Simulation in Matlab