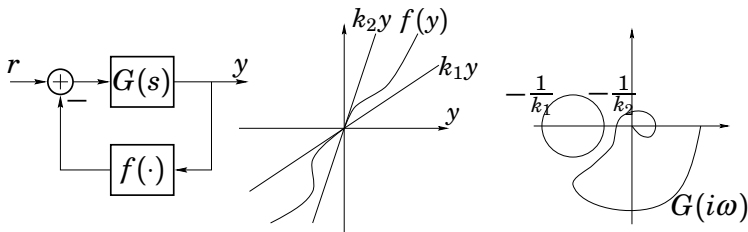


Lecture 5 — Input-output stability

or

“How to make a circle out of the point $-1 + 0i$, and different ways to stay away from it ...”



Course Outline

- | | |
|--------------|---|
| Lecture 1-3 | Modelling and basic phenomena
(linearization, phase plane, limit cycles) |
| Lecture 4-6 | Analysis methods
(Lyapunov, circle criterion, describing functions) |
| Lecture 7-8 | Common nonlinearities
(Saturation, friction, backlash, quantization) |
| Lecture 9-13 | Design methods
(Lyapunov methods, Backstepping, Optimal control) |
| Lecture 14 | Summary |

Today's Goal

To understand

- ▶ *signal norms*
- ▶ *system gain*
- ▶ *bounded input bounded output (BIBO) stability*

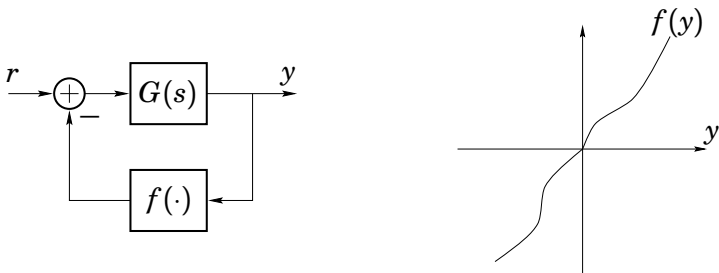
To be able to analyze stability using

- ▶ *the Small Gain Theorem,*
- ▶ *the Circle Criterion,*
- ▶ *Passivity*

Material

- ▶ *[Glad & Ljung]:* Ch 1.5-1.6, 12.3 *[Khalil]:* Ch 5–7.1
- ▶ lecture slides

History

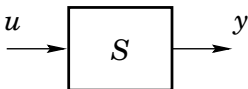


For what $G(s)$ and $f(\cdot)$ is the closed-loop system stable?

- ▶ Lur'e and Postnikov's problem (1944)
- ▶ Aizerman's conjecture (1949) (False!)
- ▶ Kalman's conjecture (1957) (False!)
- ▶ Solution by Popov (1960) (Led to the Circle Criterion)

Gain

Idea: Generalize static gain to nonlinear dynamical systems



The gain γ of S measures the largest amplification from u to y

Here S can be a constant, a matrix, a linear time-invariant system, a nonlinear system, etc

Question: How should we measure the size of u and y ?

Norms

A norm $\| \cdot \|$ measures size.

A **norm** is a function from a space Ω to $\mathbf{R}^+$, such that for all $x, y \in \Omega$

- ▶ $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
- ▶ $\|x + y\| \leq \|x\| + \|y\|$
- ▶ $\|\alpha x\| = |\alpha| \cdot \|x\|$, for all $\alpha \in \mathbf{R}$

Examples

Euclidean norm: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$

Max norm: $\|x\| = \max\{|x_1|, \dots, |x_n|\}$

Signal Norms

A signal $x(t)$ is a function from $\mathbf{R}^+$ to $\mathbf{R}^d$.

A signal norm is a way to measure the size of $x(t)$.

Examples

2-norm (energy norm): $\|x\|_2 = \sqrt{\int_0^\infty |x(t)|^2 dt}$

sup-norm: $\|x\|_\infty = \sup_{t \in \mathbf{R}^+} |x(t)|$

The space of signals with $\|x\|_2 < \infty$ is denoted $\mathcal{L}_2$.

Parseval's Theorem

Theorem If $x, y \in \mathcal{L}_2$ have the Fourier transforms

$$X(i\omega) = \int_0^\infty e^{-i\omega t} x(t) dt, \quad Y(i\omega) = \int_0^\infty e^{-i\omega t} y(t) dt,$$

then

$$\int_0^\infty y^T(t)x(t)dt = \frac{1}{2\pi} \int_{-\infty}^\infty Y^*(i\omega)X(i\omega)d\omega.$$

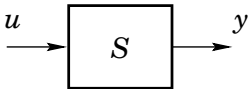
In particular

$$\|x\|_2^2 = \int_0^\infty |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty |X(i\omega)|^2 d\omega.$$

$\|x\|_2 < \infty$ corresponds to bounded energy.

System Gain

A system S is a map between two signal spaces: $y = S(u)$.



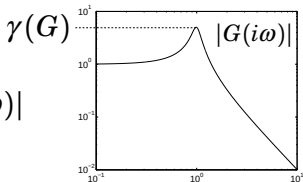
The gain of S is defined as
$$\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2} \frac{\|S(u)\|_2}{\|u\|_2}$$

Example The gain of a static relation $y(t) = \alpha u(t)$ is

$$\gamma(\alpha) = \sup_{u \in \mathcal{L}_2} \frac{\|\alpha u\|_2}{\|u\|_2} = \sup_{u \in \mathcal{L}_2} \frac{|\alpha| \|u\|_2}{\|u\|_2} = |\alpha|$$

Example—Gain of a Stable Linear System

$$\gamma(G) = \sup_{u \in \mathcal{L}_2} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{\omega \in (0, \infty)} |G(i\omega)|$$

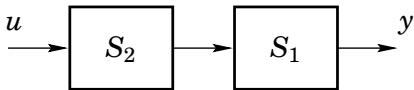


Proof: Assume $|G(i\omega)| \leq K$ for $\omega \in (0, \infty)$. Parseval's theorem gives

$$\begin{aligned} \|y\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega)|^2 |U(i\omega)|^2 d\omega \leq K^2 \|u\|_2^2 \end{aligned}$$

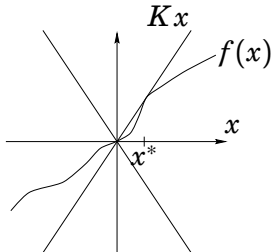
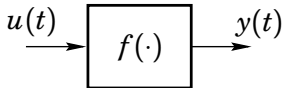
This proves that $\gamma(G) \leq K$. See [Khalil, Appendix C.10] for a proof of the equality.

2 minute exercise: Show that $\gamma(S_1 S_2) \leq \gamma(S_1) \gamma(S_2)$.



Example—Gain of a Static Nonlinearity

$$|f(x)| \leq K|x|, \quad f(x^*) = Kx^*$$

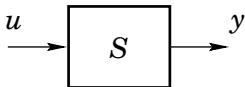


$$\|y\|_2^2 = \int_0^\infty f^2(u(t))dt \leq \int_0^\infty K^2 u^2(t)dt = K^2 \|u\|_2^2$$

$u(t) = x^*, t \in (0, \infty)$ gives equality $\Rightarrow$

$$\gamma(f) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2} = K.$$

BIBO Stability



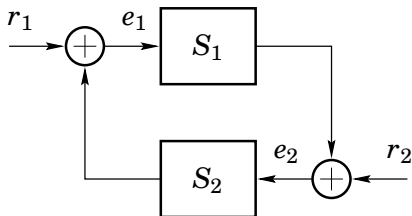
$$\gamma(S) = \sup_{u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2}$$

Definition

S is bounded-input bounded-output (BIBO) stable if $\gamma(S) < \infty$.

Example: If $\dot{x} = Ax$ is asymptotically stable then $G(s) = C(sI - A)^{-1}B + D$ is BIBO stable.

The Small Gain Theorem



Theorem

Assume S_1 and S_2 are BIBO stable. If

$$\gamma(S_1)\gamma(S_2) < 1$$

then the closed-loop map from (r_1, r_2) to (e_1, e_2) is BIBO stable.

“Proof” of the Small Gain Theorem

Existence of solution (e_1, e_2) for every (r_1, r_2) has to be verified separately. Then

$$\|e_1\|_2 \leq \|r_1\|_2 + \gamma(S_2)[\|r_2\|_2 + \gamma(S_1)\|e_1\|_2]$$

gives

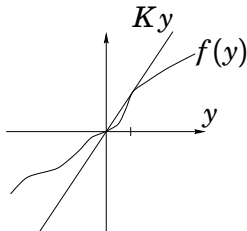
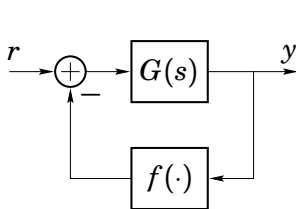
$$\|e_1\|_2 \leq \frac{\|r_1\|_2 + \gamma(S_2)\|r_2\|_2}{1 - \gamma(S_2)\gamma(S_1)}$$

$\gamma(S_2)\gamma(S_1) < 1$, $\|r_1\|_2 < \infty$, $\|r_2\|_2 < \infty$ give $\|e_1\|_2 < \infty$.
Similarly we get

$$\|e_2\|_2 \leq \frac{\|r_2\|_2 + \gamma(S_1)\|r_1\|_2}{1 - \gamma(S_1)\gamma(S_2)}$$

so also e_2 is bounded.

Linear System with Static Nonlinear Feedback (1)

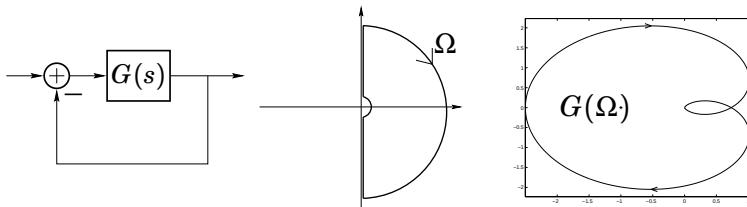


$$G(s) = \frac{2}{(s+1)^2} \quad \text{and} \quad 0 \leq \frac{f(y)}{y} \leq K$$

$$\gamma(G) = 2 \text{ and } \gamma(f) \leq K.$$

The small gain theorem gives that $K \in [0, 1/2)$ implies BIBO stability.

The Nyquist Theorem

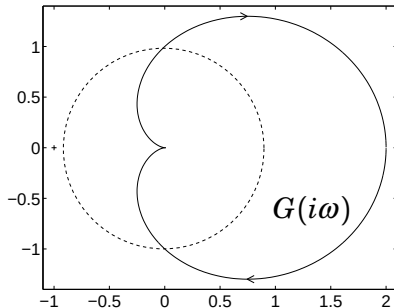


Theorem

The closed loop system is stable iff the number of counter-clockwise encirclements of -1 by $G(\Omega)$ (note: ω increasing) equals the number of open loop unstable poles.

The Small Gain Theorem can be Conservative

Let $f(y) = Ky$ for the previous system.

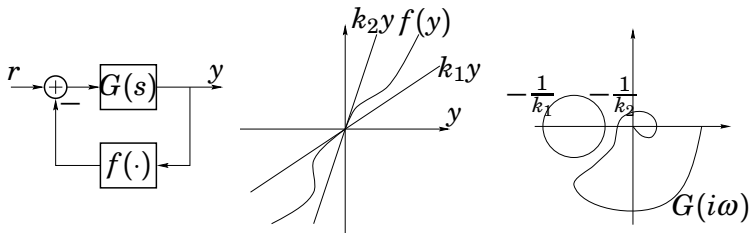


The Nyquist Theorem proves stability when $K \in [0, \infty)$.

The Small Gain Theorem proves stability when $K \in [0, 1/2)$.

The Circle Criterion

Case 1: $0 < k_1 \leq k_2 < \infty$



Theorem Consider a feedback loop with $y = Gu$ and $u = -f(y) + r$. Assume $G(s)$ is stable and that

$$0 < k_1 \leq \frac{f(y)}{y} \leq k_2.$$

If the Nyquist curve of $G(s)$ does not intersect or encircle the circle defined by the points $-1/k_1$ and $-1/k_2$, then the closed-loop system is BIBO stable from r to y .

Other cases

G : stable system

- ▶ $0 < k_1 < k_2$: Stay outside circle
- ▶ $0 = k_1 < k_2$: Stay to the right of the line $\text{Re } s = -1/k_2$
- ▶ $k_1 < 0 < k_2$: Stay inside the circle

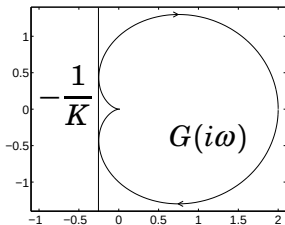
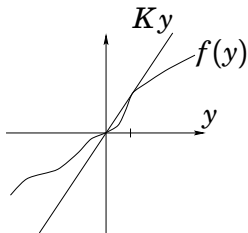
Other cases: Multiply f and G with -1 .

G : Unstable system

To be able to guarantee stability, k_1 and k_2 must have same sign (otherwise unstable for $k = 0$)

- ▶ $0 < k_1 < k_2$: Encircle the circle p times counter-clockwise (if ω increasing)
- ▶ $k_1 < k_2 < 0$: Encircle the circle p times counter-clockwise (if ω increasing)

Linear System with Static Nonlinear Feedback (2)



The “circle” is defined by $-1/k_1 = -\infty$ and $-1/k_2 = -1/K$.

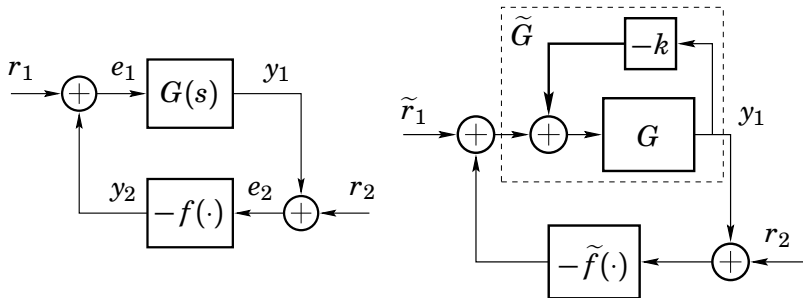
$$\min \text{Re } G(i\omega) = -1/4$$

so the Circle Criterion gives that if $K \in [0, 4)$ the system is BIBO stable.

Proof of the Circle Criterion

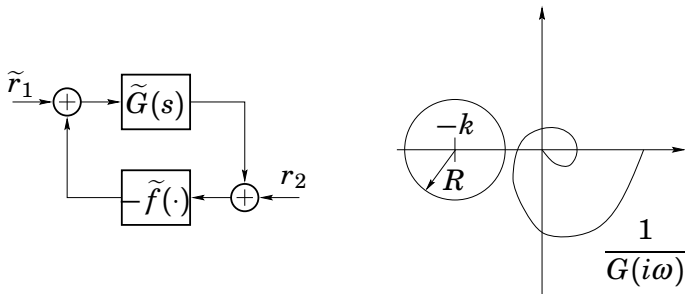
Let $k = (k_1 + k_2)/2$ and $\tilde{f}(y) = f(y) - ky$. Then

$$\left| \frac{\tilde{f}(y)}{y} \right| \leq \frac{k_2 - k_1}{2} =: R$$



$$\tilde{r}_1 = r_1 - kr_2$$

Proof of the Circle Criterion (cont'd)



SGT gives stability for $|\tilde{G}(i\omega)|R < 1$ with $\tilde{G} = \frac{G}{1 + kG}$.

$$R < \frac{1}{|\tilde{G}(i\omega)|} = \left| \frac{1}{G(i\omega)} + k \right|$$

Transform this expression through $z \rightarrow 1/z$.

Lyapunov revisited

Original idea: “Energy is decreasing”

$$\begin{aligned}\dot{x} &= f(x), & x(0) &= x_0 \\ V(x(T)) - V(x(0)) &\leq 0 \\ &(+\text{some other conditions on } V)\end{aligned}$$

New idea: “Increase in stored energy $\leq$ added energy”

$$\begin{aligned}\dot{x} &= f(x, u), & x(0) &= x_0 \\ y &= h(x)\end{aligned}$$
$$V(x(T)) - V(x(0)) \leq \int_0^T \underbrace{\varphi(y, u)}_{\text{external power}} dt \quad (1)$$

Motivation

Will assume the external power has the form $\phi(y, u) = y^T u$.

Only interested in BIBO behavior. Note that

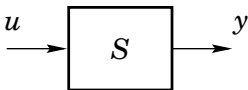
$$\exists V \geq 0 \text{ with } V(x(0)) = 0 \text{ and (1)}$$

$$\iff$$

$$\int_0^T y^T u \, dt \geq 0$$

Motivated by this we make the following definition

Passive System



Definition The system S is **passive** from u to y if

$$\int_0^T y^T u \, dt \geq 0, \quad \text{for all } u \text{ and all } T > 0$$

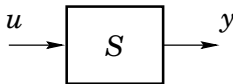
and **strictly passive** from u to y if there $\exists \epsilon > 0$ such that

$$\int_0^T y^T u \, dt \geq \epsilon(|y|_T^2 + |u|_T^2), \quad \text{for all } u \text{ and all } T > 0$$

A Useful Notation

Define the **scalar product**

$$\langle y, u \rangle_T = \int_0^T y^T(t) u(t) dt$$

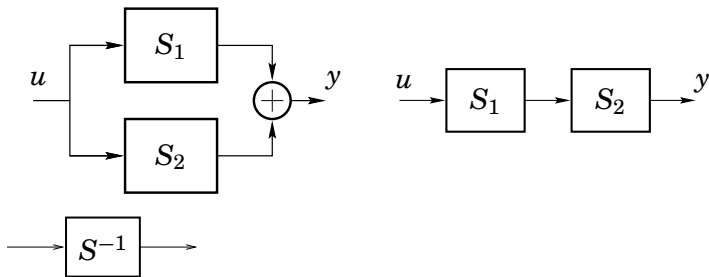


Cauchy-Schwarz inequality:

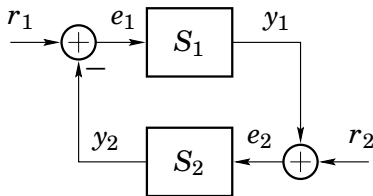
$$\langle y, u \rangle_T \leq |y|_T |u|_T$$

where $|y|_T = \sqrt{\langle y, y \rangle_T}$. Note that $|y|_\infty = \|y\|_2$.

2 minute exercise:



Feedback of Passive Systems is Passive



If S_1 and S_2 are passive, then the closed-loop system from (r_1, r_2) to (y_1, y_2) is also passive.

Proof:

$$\begin{aligned}\langle y, r \rangle_T &= \langle y_1, r_1 \rangle_T + \langle y_2, r_2 \rangle_T \\ &= \langle y_1, r_1 - y_2 \rangle_T + \langle y_2, r_2 + y_1 \rangle_T \\ &= \langle y_1, e_1 \rangle_T + \langle y_2, e_2 \rangle_T \geq 0\end{aligned}$$

Hence, $\langle y, r \rangle_T \geq 0$ if $\langle y_1, e_1 \rangle_T \geq 0$ and $\langle y_2, e_2 \rangle_T \geq 0$

Passivity of Linear Systems

Theorem An asymptotically stable linear system $G(s)$ is **passive** if and only if

$$\operatorname{Re} G(i\omega) \geq 0, \quad \forall \omega > 0$$

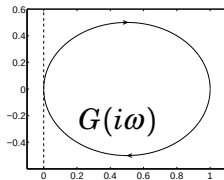
It is **strictly passive** if and only if there exists $\epsilon > 0$ such that

$$\operatorname{Re} G(i\omega) \geq \epsilon(1 + |G(i\omega)|^2), \quad \forall \omega > 0$$

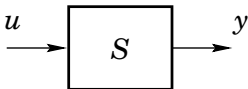
Example

$G(s) = \frac{s+1}{s+2}$ is passive and strictly passive,

$G(s) = \frac{1}{s}$ is passive but not strictly passive.



A Strictly Passive System Has Finite Gain



If S is strictly passive, then $\gamma(S) < \infty$.

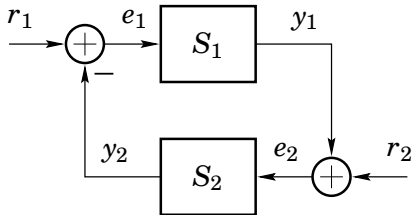
Proof: Note that $\|y\|_2 = \lim_{T \rightarrow \infty} |y|_T$.

$$\epsilon(|y|_T^2 + |u|_T^2) \leq \langle y, u \rangle_T \leq |y|_T \cdot |u|_T \leq \|y\|_2 \cdot \|u\|_2$$

Hence, $\epsilon|y|_T^2 \leq \|y\|_2 \cdot \|u\|_2$, so letting $T \rightarrow \infty$ gives

$$\|y\|_2 \leq \frac{1}{\epsilon} \|u\|_2$$

The Passivity Theorem



Theorem If S_1 is strictly passive and S_2 is passive, then the closed-loop system is BIBO stable from r to y .

Proof of the Passivity Theorem

S_1 strictly passive and S_2 passive give

$$\epsilon(|y_1|_T^2 + |e_1|_T^2) \leq \langle y_1, e_1 \rangle_T + \langle y_2, e_2 \rangle_T = \langle y, r \rangle_T$$

Therefore

$$|y_1|_T^2 + \langle r_1 - y_2, r_1 - y_2 \rangle_T \leq \frac{1}{\epsilon} \langle y, r \rangle_T$$

or

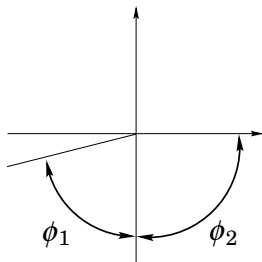
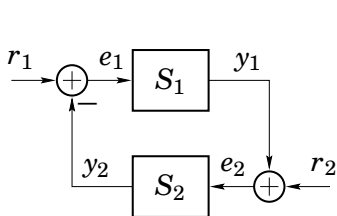
$$|y_1|_T^2 + |y_2|_T^2 - 2\langle y_2, r_2 \rangle_T + |r_1|_T^2 \leq \frac{1}{\epsilon} \langle y, r \rangle_T$$

Finally

$$|y|_T^2 \leq 2\langle y_2, r_2 \rangle_T + \frac{1}{\epsilon} \langle y, r \rangle_T \leq \left(2 + \frac{1}{\epsilon}\right) |y|_T |r|_T$$

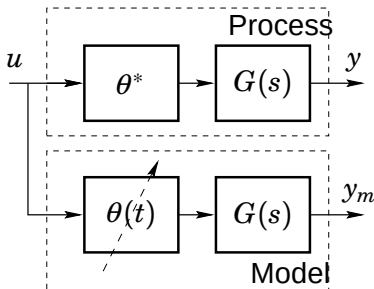
Letting $T \rightarrow \infty$ gives $\|y\|_2 \leq C\|r\|_2$ and the result follows

Passivity Theorem is a “Small Phase Theorem”



Example—Gain Adaptation

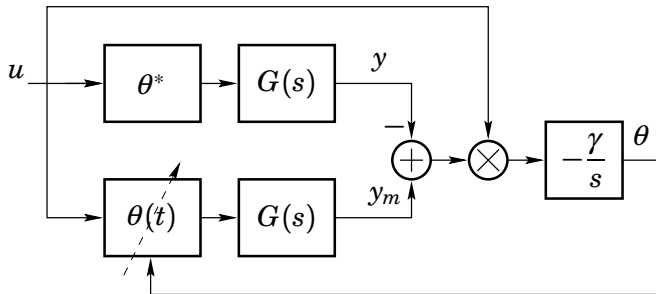
Applications in channel estimation in telecommunication, noise cancelling etc.



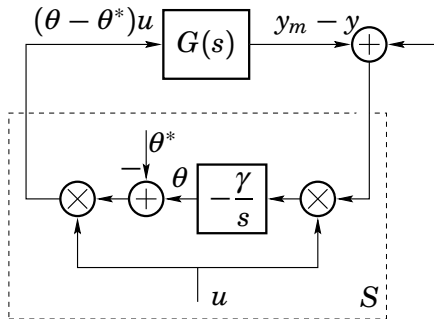
Adaptation law:

$$\frac{d\theta}{dt} = -\gamma u(t)[y_m(t) - y(t)], \quad \gamma > 0.$$

Gain Adaptation—Closed-Loop System



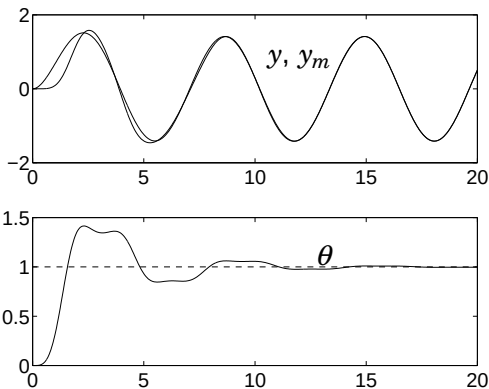
Gain Adaptation is BIBO Stable



S is passive (Exercise 4.12), so the closed-loop system is BIBO stable if $G(s)$ is strictly passive.

Simulation of Gain Adaptation

Let $G(s) = \frac{1}{s+1} + \epsilon$, $\gamma = 1$, $u = \sin t$, $\theta(0) = 0$ and $\gamma^* = 1$



Storage Function

Consider the nonlinear control system

$$\dot{x} = f(x, u), \quad y = h(x)$$

A **storage function** is a C^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- ▶ $V(0) = 0$ and $V(x) \geq 0, \quad \forall x \neq 0$
- ▶ $\dot{V}(x) \leq u^T y, \quad \forall x, u$

Remark:

- ▶ $V(T)$ represents the stored energy in the system
- ▶
$$\underbrace{V(x(T))}_{\text{stored energy at } t = T} \leq \underbrace{\int_0^T y(t)u(t)dt}_{\text{absorbed energy}} + \underbrace{V(x(0))}_{\text{stored energy at } t = 0},$$

 $\forall T > 0$

Storage Function and Passivity

Lemma: If there exists a storage function V for a system

$$\dot{x} = f(x, u), \quad y = h(x)$$

with $x(0) = 0$, then the system is passive.

Proof: For all $T > 0$,

$$\langle y, u \rangle_T = \int_0^T y(t)u(t)dt \geq V(x(T)) - V(x(0)) = V(x(T)) \geq 0$$

Lyapunov vs. Passivity

Storage function is a generalization of Lyapunov function

Lyapunov idea: “Energy is decreasing”

$$\dot{V} \leq 0$$

Passivity idea: “Increase in stored energy $\leq$ Added energy”

$$\dot{V} \leq u^T y$$

Example KYP Lemma

Consider an asymptotically stable linear system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

Assume there exists positive definite symmetric matrices P, Q such that

$$A^T P + PA = -Q, \text{ and } B^T P = C$$

Consider $V = 0.5x^T P x$. Then

$$\begin{aligned}\dot{V} &= 0.5(\dot{x}^T P x + x^T P \dot{x}) = 0.5x^T (A^T P + PA)x + u^T B^T P x \\ &= -0.5x^T Q x + u^T y < u^T y, \quad x \neq 0\end{aligned}\tag{2}$$

and hence the system is strictly passive. This fact is part of the Kalman-Yakubovich-Popov lemma.

Next Lecture

- ▶ Describing functions (analysis of oscillations)