

# Lecture 3

- Phase-plane analysis
- Classification of singularities
- Stability of periodic solutions

## Material

- Glad and Ljung: Chapter 13
- Khalil: Chapter 2.1–2.3
- Lecture notes

# Today's Goal

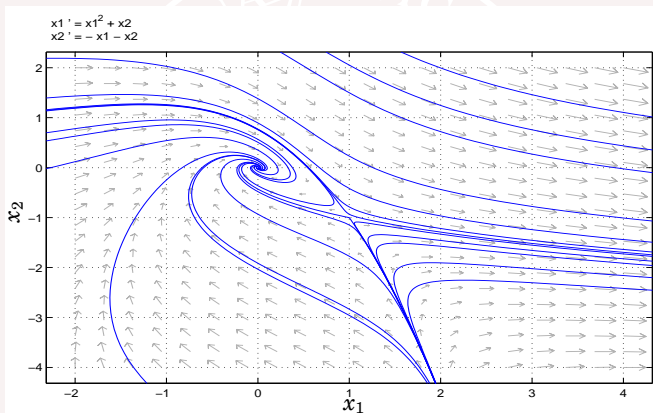
*You should be able to*

- *sketch phase portraits for two-dimensional systems*
- *classify equilibria into nodes, focus, saddle points, and center points.*
- *analyze limit cycles through Poincaré maps*

## First glimpse of phase plane portraits: Consider the system

$$\dot{x}_1 = x_1^2 + x_2$$

$$\dot{x}_2 = -x_1 - x_2$$

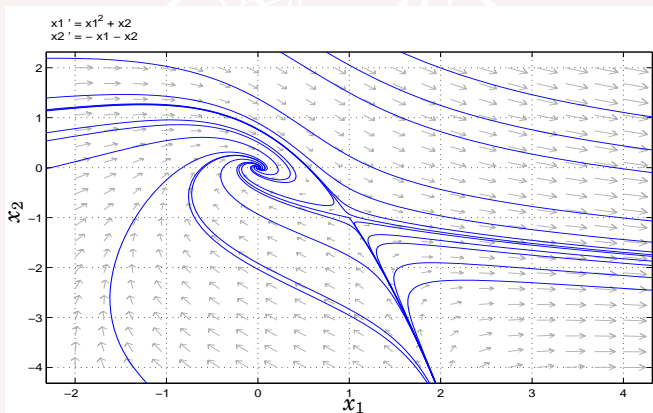


Flow-interpretation: To each point  $(x_1, x_2)$  in the plane there is an associated flow-direction  $\frac{dx}{dt} = f(x_1, x_2)$

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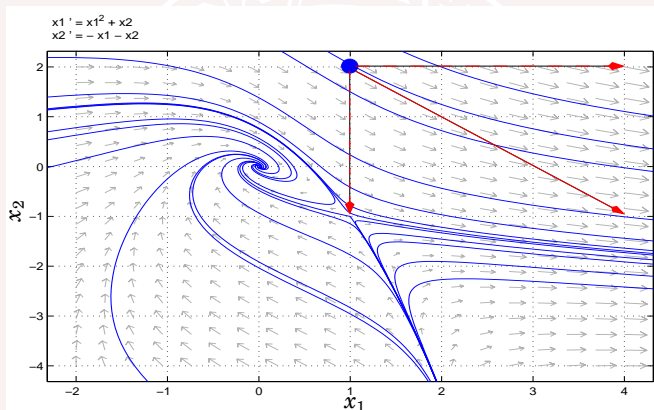


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## First glimpse of phase plane portraits: Consider the system

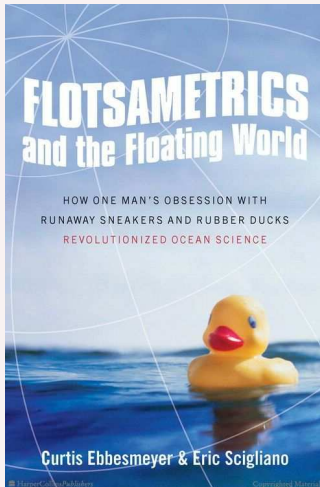
$$\dot{x}_1 = x_1^2 + x_2$$

$$\dot{x}_2 = -x_1 - x_2$$

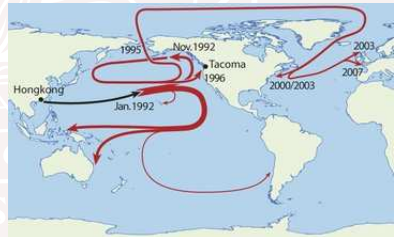


In the point  $(x_1, x_2) = (1, 2)$  the **vector field** is pointing in the direction  $(1^2 + 2, -1 - 2) = (3, -3)$ .

# Vectorfields in Oceanography ...



*On 10 January 1992, during a storm in the North Pacific Ocean close to the International Date Line, twelve 40-foot (13.3 m) intermodal containers were washed overboard. One of these containers held 28,800 Floatees,...*



[http://en.wikipedia.org/wiki/Friendly\\_Floatees](http://en.wikipedia.org/wiki/Friendly_Floatees)

# Linear Systems Revival

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analytic solution:  $x(t) = e^{At}x(0)$ .

If  $A$  is diagonalizable, then

$$e^{At} = Ve^{At}V^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where  $v_1, v_2$  are the eigenvectors of  $A$  ( $Av_1 = \lambda_1 v_1$  etc).

Matlab:

```
» [V,Lambda]=eig(A)
```

## Example: Two real negative eigenvalues

Given the eigenvalues  $\underbrace{\lambda_1}_{\text{faster}} < \underbrace{\lambda_2}_{\text{slower}} < 0$ , with corresponding eigenvectors  $v_1$  and  $v_2$ , respectively.

Solution:  $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

*Fast eigenvalue/vector:*  $x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$  for small  $t$ .  
Moves along the fast eigenvector for small  $t$

*Slow eigenvalue/vector:*  $x(t) \approx c_2 e^{\lambda_2 t} v_2$  for large  $t$ .  
Moves along the slow eigenvector towards  $x = 0$  for large  $t$

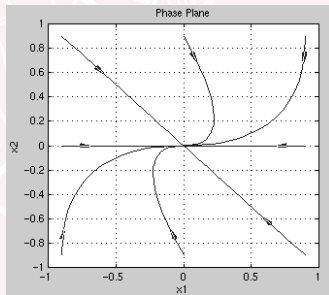


## Example—Stable Node

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$$(\lambda_1, \lambda_2) = (-1, -2) \quad \text{and} \quad [v_1 \ v_2] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$v_1$  is the slow direction and  $v_2$  is the fast.



# Equilibrium Points for Linear Systems

$\text{Im}\lambda_i = 0$  :      stable node  
 $\lambda_1, \lambda_2 < 0$

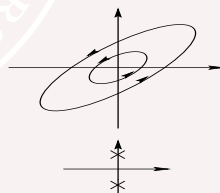
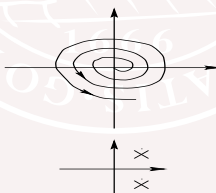
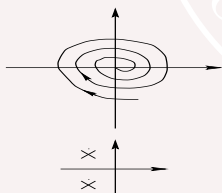
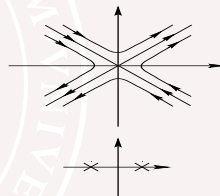
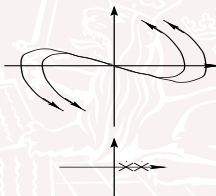
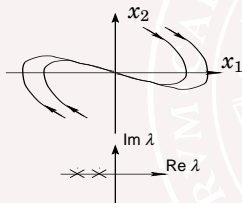
unstable node  
 $\lambda_1, \lambda_2 > 0$

saddle point  
 $\lambda_1 < 0 < \lambda_2$

$\text{Im}\lambda_i \neq 0$  :       $\text{Re}\lambda_i < 0$   
 stable focus

$\text{Re}\lambda_i > 0$   
 unstable focus

$\text{Re}\lambda_i = 0$   
 center point



## Example—Unstable Focus

$$\dot{x} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x, \quad \sigma, \omega > 0, \quad \lambda_{1,2} = \sigma \pm i\omega$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\sigma t} e^{i\omega t} & 0 \\ 0 & e^{\sigma t} e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} x(0)$$

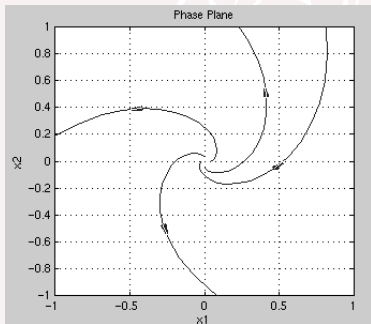
In polar coordinates  $r = \sqrt{x_1^2 + x_2^2}$ ,  $\theta = \arctan x_2/x_1$   
( $x_1 = r \cos \theta$ ,  $x_2 = r \sin \theta$ ):

$$\dot{r} = \sigma r$$

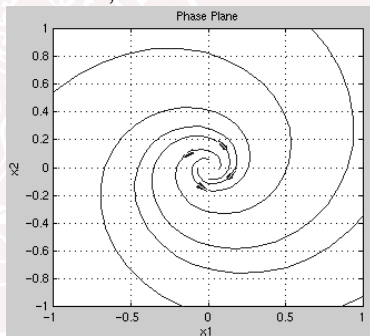
$$\dot{\theta} = \omega$$

## Example- unstable focus cont'd

$$\lambda_{1,2} = 1 \pm i$$



$$\lambda_{1,2} = 0.3 \pm i$$



# Equilibrium Points for Linear Systems

$\text{Im}\lambda_i = 0$  :      stable node  
 $\lambda_1, \lambda_2 < 0$

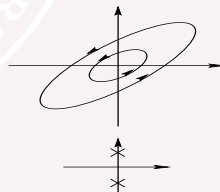
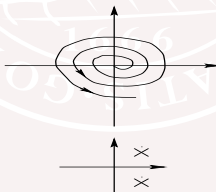
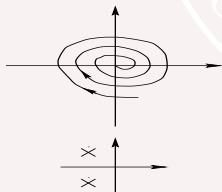
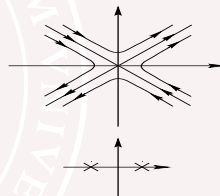
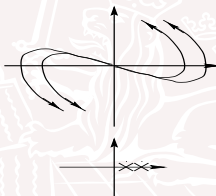
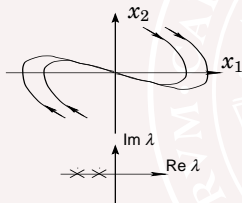
unstable node  
 $\lambda_1, \lambda_2 > 0$

saddle point  
 $\lambda_1 < 0 < \lambda_2$

$\text{Im}\lambda_i \neq 0$  :       $\text{Re}\lambda_i < 0$   
 stable focus

$\text{Re}\lambda_i > 0$   
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$\text{Re}\lambda_i = 0$   
 center point



## 4 minute exercise

*What is the phase portrait if  $\lambda_1 = \lambda_2$ ?*

*Hint:* For  $\lambda_1 = \lambda_2 = \lambda$  there are two different cases: only one linearly independent eigenvector or all vectors are eigenvectors

# Star Node or Multi-Tangent Node

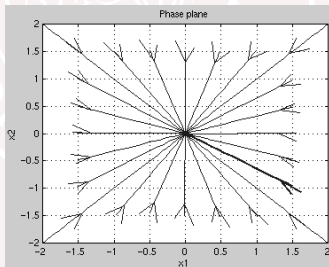
Case I: If

$$\dot{x} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} x, \quad \text{rank}(\lambda I - A) = 0$$

then the solution is

$$x_1(t) = x_1(0)e^{\lambda t}$$

$$x_2(t) = x_2(0)e^{\lambda t}$$



# One Tangent Node

Case II: If

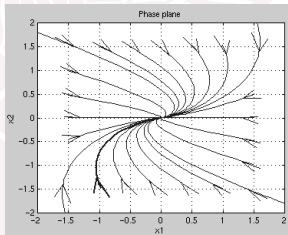
$$\dot{x} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} x, \quad \text{rank}(\lambda I - A) = 1$$

then the solution is

$$x_1(t) = x_1(0)e^{\lambda t} + tx_2(0)e^{\lambda t}$$

$$x_2(t) = x_2(0)e^{\lambda t}$$

There is only one eigenvector:  $v_1 = \alpha v_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ .





# Linear *Time-Varying* Systems (warning)

**Warning:** Pointwise “Left Half-Plane eigenvalues” of  $A(t)$  (i.e., *time-varying systems*) do *NOT* impose stability!!!

$$A(t) = \begin{pmatrix} -1 + \alpha \cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Pointwise eigenvalues are given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

which are in the LHP for  $0 < \alpha < 2$  (and here even constant). However,

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t} \cos t & e^{-t} \sin t \\ -e^{(\alpha-1)t} \sin t & e^{-t} \cos t \end{pmatrix} x(0),$$

which is an unbounded solution for  $\alpha > 1$ .

# Phase-Plane Analysis for Nonlinear Systems

Close to equilibria “nonlinear system”  $\approx$  “linear system”.

**Theorem** Assume

$$\dot{x} = f(x)$$

is linearized at  $x_0$  so that

$$\dot{x} = Ax + g(x),$$

where  $g \in C^1$  and  $\frac{g(x)-g(x_0)}{\|x-x_0\|} \rightarrow 0$  as  $x \rightarrow x_0$ .

If  $\dot{z} = Az$  has a focus, node, or saddle point, then  $\dot{x} = f(x)$  has the same type of equilibrium at the origin.

If the linearized system has a center, then the nonlinear system has either a center or a focus.

# How to Draw Phase Portraits

If done by hand then

- 1 Find equilibria (also called singularities)
- 2 Sketch local behavior around equilibria
- 3 Sketch  $(\dot{x}_1, \dot{x}_2)$  for some other points. Use that  $\frac{dx_1}{dx_2} = \frac{\dot{x}_1}{\dot{x}_2}$ .
- 4 Try to find possible limit cycles
- 5 Guess solutions

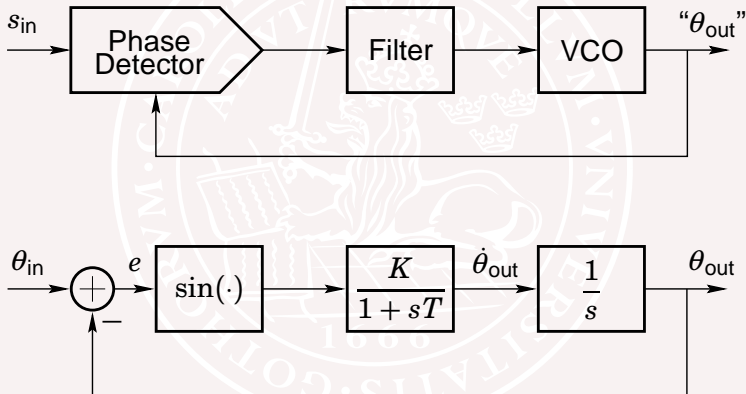
Matlab: pptool6/pptool7, dfield6/dfield7, dee, ICTools, etc.

PPTool and some other tools for Matlab is available on or via

<http://www.control.lth.se/course/FRTN05>

# Phase-Locked Loop

A PLL tracks phase  $\theta_{in}(t)$  of a signal  $s_{in}(t) = A \sin[\omega t + \theta_{in}(t)]$ .



# Singularity Analysis of PLL

Let  $x_1(t) = \theta_{\text{out}}(t)$  and  $x_2(t) = \dot{\theta}_{\text{out}}(t)$ .

Assume  $K, T > 0$  and  $\theta_{\text{in}}(t) = \theta_{\text{in}}$  constant.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -T^{-1}x_2 + KT^{-1}\sin(\theta_{\text{in}} - x_1)$$

Singularities are  $(\theta_{\text{in}} + n\pi, 0)$ , since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$

$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{\text{in}} - x_1) = 0 \Rightarrow x_1 = \theta_{\text{in}} + n\pi$$

# Singularity Classification of Linearized System

Linearization gives the following characteristic equations:

$n$  even:

$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

$K > (4T)^{-1}$  gives stable focus

$0 < K < (4T)^{-1}$  gives stable node

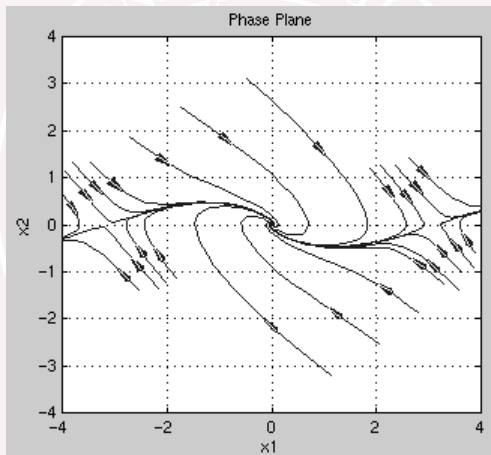
$n$  odd:

$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all  $K, T > 0$

# Phase-Plane for PLL

$K = 1/2, T = 1$ : Focus  $(2k\pi, 0)$ , saddle points  $((2k + 1)\pi, 0)$



# Summary

Phase-plane analysis limited to second-order systems  
(sometimes it is possible for higher-order systems to fix some states)

Many dynamical systems of order three and higher not fully understood (chaotic behaviors etc.)



## Bonus — Discrete Time

Many results are parallel (observability, controllability,...)

Example: The difference equation

$$x_{k+1} = f(x_k)$$

is asymptotically stable at  $x^*$  if the linearization

$$\left. \frac{\partial f}{\partial x} \right|_{x^*} \text{ has all eigenvalues in } |\lambda| < 1$$

(that is, within the unit circle).

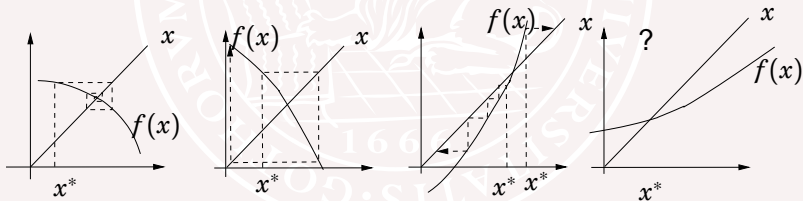
## Example (cont'd): Numerical iteration

$$x_{k+1} = f(x_k)$$

to find fixed point

$$x^* = f(x^*)$$

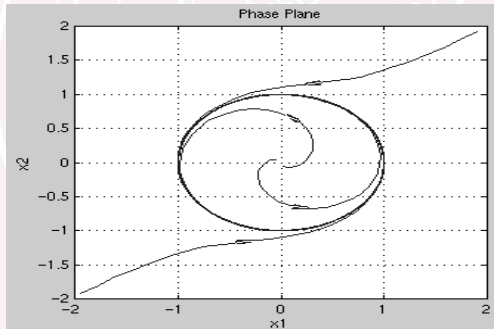
When does the iteration converge?



# Periodic Solutions: $x(t + T) = x(t)$

Example of an asymptotically stable periodic solution:

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}\tag{1}$$



# Periodic solution: Polar coordinates.

Let

$$x_1 = r \cos \theta \quad \Rightarrow \quad dx_1 = \cos \theta dr - r \sin \theta d\theta$$

$$x_2 = r \sin \theta \quad \Rightarrow \quad dx_2 = \sin \theta dr + r \cos \theta d\theta$$

$\Rightarrow$

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now

$$\dot{x}_1 = r(1 - r^2) \cos \theta - r \sin \theta$$

$$\dot{x}_2 = r(1 - r^2) \sin \theta + r \cos \theta$$

which gives

$$\dot{r} = r(1 - r^2)$$

$$\dot{\theta} = 1$$

Only  $r = 1$  is a stable equilibrium!

A system has a **periodic solution** if for some  $T > 0$

$$x(t + T) = x(t), \quad \forall t \geq 0$$

*Note* that a constant value for  $x(t)$  by convention not is regarded as periodic.

- When does a periodic solution exist?
- When is it locally (asymptotically) stable? When is it globally asymptotically stable?

# Poincaré map (“Stroboscopic map”)

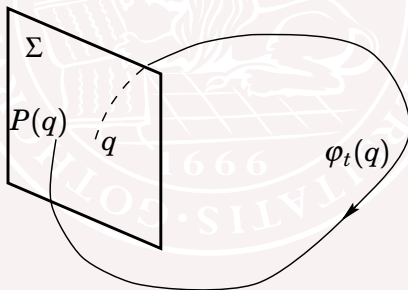
$$\dot{x} = f(x), \quad x \in \mathbf{R}^n$$

$\varphi_t(q)$  is the solution starting in  $q$  after time  $t$ .

$\Sigma \subset \mathbf{R}^{n-1}$  is a hyperplane transverse to  $\varphi_t$ .

The Poincaré map  $P : \Sigma \rightarrow \Sigma$  is

$$P(q) = \varphi_{\tau(q)}(q), \quad \tau(q) \text{ is the first return time}$$

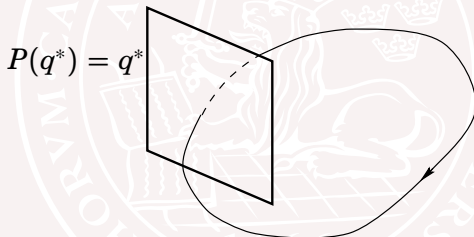


# Limit Cycles

If a simple periodic orbit pass through  $q^*$ , then  $P(q^*) = q^*$ .

Such an orbit is called a *limit cycle*.

$q^*$  is called a *fixed point* of  $P$ .



Does the iteration  $q_{k+1} = P(q_k)$  converge to  $q^*$ ?

# Locally Stable Limit Cycles

The linearization of  $P$  around  $q^*$  gives a matrix  $W = \left. \frac{\partial P}{\partial q} \right|_{q^*}$  so

$$(q_{k+1} - q^*) \approx W(q_k - q^*),$$

if  $q_k$  is close to  $q^*$ .

- If all  $|\lambda_i(W)| < 1$ , then the corresponding limit cycle is locally **asymptotically stable**.
- If  $|\lambda_i(W)| > 1$ , then the limit cycle is **unstable**.



# Linearization Around a Periodic Solution

The linearization of

$$\dot{x}(t) = f(x(t))$$

around  $x_0(t) = x_0(t + T)$  is

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$$

$$A(t) = \frac{\partial f}{\partial x}(x_0(t)) = A(t + T)$$

$P$  is the map from the solution at  $t = 0$  to  $t = \tau(q)$ .

## Example—Stable Unit Circle

Rewrite (1) in polar coordinates:

$$\dot{r} = r(1 - r^2)$$

$$\dot{\theta} = 1$$

Choose  $\Sigma = \{(r, \theta) : r > 0, \theta = 2\pi k\}$ .

The solution is

$$\varphi_t(r_0, \theta_0) = \left( [1 + (r_0^{-2} - 1)e^{-2t}]^{-1/2}, t + \theta_0 \right)$$

First return time from any point  $(r_0, \theta_0) \in \Sigma$  is  $\tau(r_0, \theta_0) = 2\pi$ .

## Example—Stable Unit Circle

The Poincaré map is

$$P(r_0) = [1 + (r_0^{-2} - 1)e^{-2 \cdot 2\pi}]^{-1/2}$$

$r_0 = 1$  is a fixed point.

The limit cycle that corresponds to  $r(t) = 1$  and  $\theta(t) = t$  is locally asymptotically stable, because

$$W = \frac{dP}{dr_0}(1) = [e^{-4\pi}]$$

and

$$|W| = \left| \frac{dP}{dr_0}(1) \right| = |e^{-4\pi}| < 1$$

## Example—The Hand Saw

Can we stabilize the inverted pendulum by vertical oscillations?



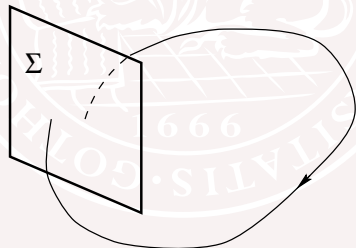
# The Hand Saw—Poincaré Map

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{\ell} \left( g + a\omega^2 \sin x_3 \right) \sin x_1$$

$$\dot{x}_3(t) = \omega$$

Choose  $\Sigma = \{x_3 = 2\pi k\}$ .



# The Hand Saw–Poincaré Map

$q^* = 0$  and  $T = 2\pi/\omega$ . No explicit expression for  $P$ . It is, however, easy to determine  $W$  numerically. Do two (or *preferably many more*) different simulations with different, small, initial conditions  $x(0) = y$  and  $x(0) = z$ . Solve  $W$  through (*least squares solution of*)

$$\begin{pmatrix} x(T) \big|_{x(0)=y} & x(T) \big|_{x(0)=z} \end{pmatrix} = W \begin{pmatrix} y & z \end{pmatrix}$$

This gives for  $a = 1\text{cm}$ ,  $\ell = 17\text{cm}$ ,  $\omega = 180$

$$W = \begin{pmatrix} 1.37 & 0.035 \\ -3.86 & 0.630 \end{pmatrix}$$

which has eigenvalues  $(1.047, 0.955)$ . Unstable.

$W$  is stable for  $\omega > 183$

# The Hand Saw—Stability Condition

Make the assumptions that

$$\ell \gg a \quad \text{and} \quad a\omega^2 \gg g$$

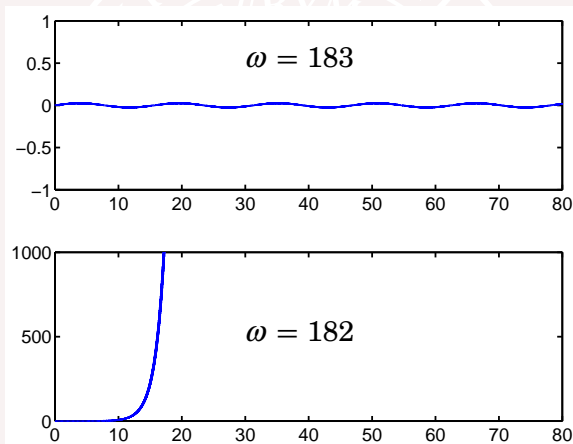
Then some calculations show that the Poincaré map is stable at  $q^* = 0$  when

$$\omega > \frac{\sqrt{2g\ell}}{a}$$

$a = 1$  cm and  $\ell = 17$  cm give  $\omega > 182.6$  rad/s (29 Hz).

# The Hand Saw—Simulation

Simulation results give good agreement



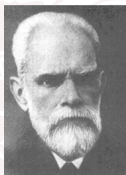


# Next Lecture

- Lyapunov methods for stability analysis

Lyapunov generalized the idea of: *If the total energy is dissipated along the trajectories (i.e the solution curves), the system must be stable.*

Benefit: Might conclude that a system is stable or asymptotically stable **without solving** the nonlinear differential equation.



Nonlinear control is a serious business... cheer up ☺

# Lab 1

