

# Lecture 14 — Course Summary

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- ▶ CEQ
- ▶ The exam
- ▶ Questions / review of the course

## CEQ

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You will get a mail regarding CEQ (Course evaluation) to be filled out via a web-page.

Please, fill it in, and **write your comments**.

Both Swedish and English versions are available!

Remember, without your feedback we teach in open-loop.

## Question: What's on the exam?

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Among old exam questions:

- ▶ Models, equilibria etc
- ▶ Linearization and stability
- ▶ Circle criterion
- ▶ Small gain
- ▶ Describing Functions
- ▶ Lyapunov functions
- ▶ ...

Old exams and solutions are available from the course home page.

# Exam (in English)

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## Course Material Allowed:

- ▶ Lecture slides 1-14 (no notes, no exercises or old exams)
- ▶ Laboratory exercises 1, 2, and 3
- ▶ *Reglerteori* by Glad and Ljung
- ▶ *Applied Nonlinear Control* by Slotine and Li
- ▶ *Nonlinear Systems* by Khalil

You may bring everything on the list + “Collection of Formulae for Control” to the exam.

## Question

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Can I get different answers if use the Small Gain theorem and the Circle criterion? What does it mean?

- ▶ If the conditions for stability are not satisfied for one criterion it does not necessarily mean that the system is unstable. It just means that you can not use that method to guarantee stability. You can never 'prove' that a system is stable with one method and 'unstable' with another.
- ▶ Similarly, there are no general guaranteed methods to find a Lyapunov function (though some suggested good methods/candidates are worth to try).

# Question

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Please repeat the stability definitions and methods to prove stability.

Explain about invariant sets and when  $\dot{V} = 0$ .

# Stability Definitions

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An equilibrium point  $x = 0$  of  $\dot{x} = f(x)$  is

**locally stable**, if for every  $R > 0$  there exists  $r > 0$ , such that

$$\|x(0)\| < r \quad \Rightarrow \quad \|x(t)\| < R, \quad t \geq 0$$

**locally asymptotically stable**, if locally stable and

$$\|x(0)\| < r \quad \Rightarrow \quad \lim_{t \rightarrow \infty} x(t) = 0$$

**globally asymptotically stable**, if asymptotically stable for all  $x(0) \in \mathbf{R}^n$ .

# Lyapunov Theorem for Local Stability

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**Theorem** Let  $\dot{x} = f(x)$ ,  $f(0) = 0$ , and  $0 \in \Omega \subset \mathbf{R}^n$  for some open set  $\Omega$ . Assume that  $V : \Omega \rightarrow \mathbf{R}$  is a  $C^1$  function. If

- ▶  $V(0) = 0$
- ▶  $V(x) > 0$ , for all  $x \in \Omega$ ,  $x \neq 0$
- ▶  $\dot{V}(x) \leq 0$  along all trajectories in  $\Omega$

then  $x = 0$  is locally stable. Furthermore, if also

- ▶  $\dot{V}(x) < 0$  for all  $x \in \Omega$ ,  $x \neq 0$

then  $x = 0$  is locally asymptotically stable.



# Lyapunov Theorem for Global Stability

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**Theorem** Let  $\dot{x} = f(x)$  and  $f(0) = 0$ . Assume that  $V : \mathbf{R}^n \rightarrow \mathbf{R}$  is a  $C^1$  function. If

- ▶  $V(0) = 0$
- ▶  $V(x) > 0$ , for all  $x \neq 0$
- ▶  $\dot{V}(x) < 0$  for all  $x \neq 0$
- ▶  $V(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$  (radial unboundedness)

then  $x = 0$  is globally asymptotically stable.

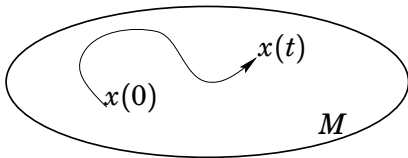
# Invariant Sets

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**Definition** A set  $M$  is called **invariant** if for the system

$$\dot{x} = f(x),$$

$x(0) \in M$  implies that  $x(t) \in M$  for all  $t \geq 0$ .



When finding Lyapunov function candidates with  $\frac{dV}{dt} \leq 0$  we often want to use this like to show that the origin is the largest invariant set  $M$ .

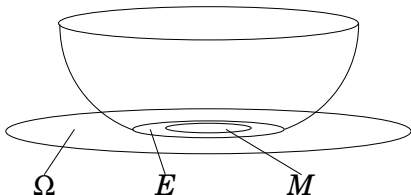
# Invariant Set Theorem

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**Theorem** Let  $\Omega \in \mathbf{R}^n$  be a bounded and closed set that is invariant with respect to

$$\dot{x} = f(x).$$

Let  $V : \mathbf{R}^n \rightarrow \mathbf{R}$  be a  $C^1$  function such that  $\dot{V}(x) \leq 0$  for  $x \in \Omega$ . Let  $E$  be the set of points in  $\Omega$  where  $\dot{V}(x) = 0$ . If  $M$  is the largest invariant set in  $E$ , then every solution with  $x(0) \in \Omega$  approaches  $M$  as  $t \rightarrow \infty$



## Example

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Example:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - x_1^3$$

Try with

$$V(x) = x_1^2 + x_2^2 \quad (\text{Alt. 1})$$

or

$$V(x) = 0.5x_1^4 + x_2^2 \quad (\text{Alt. 2})$$

Alt 1:

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V}{\partial x_2} \frac{dx_2}{dt} \\ &= 2x_1x_2 + 2x_2(-x_2 - x_1^3) = 2(x_1 - x_1^3)x_2 - 2x_2^2 \quad !??\end{aligned}$$

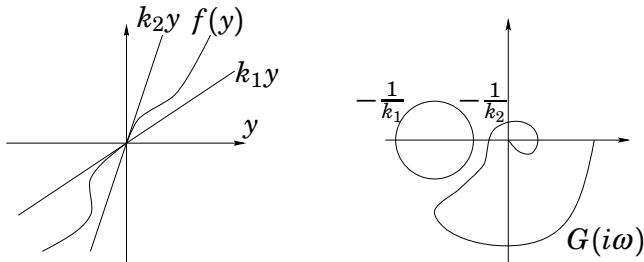
No information as indefinite.

Alt 2:

$$\frac{dV}{dt} = 0.5 \cdot 4x_1^3x_2 + 2x_2(-x_2 - x_1^3) = -2x_2^2 \leq 0$$

To show *asymptotic* stability we need to continue (Alt.2) and use LaSalle or the invariance set theorem!

## The Circle Criterion, $0 < k_1 \leq k_2 < \infty$



**Theorem** Consider a feedback loop with  $y = Gu$  and  $u = -f(y)$ . Assume  $G(s)$  is stable and that

$$k_1 \leq \frac{f(y)}{y} \leq k_2.$$

If the Nyquist curve of  $G(s)$  stays outside the circle defined by the points  $-1/k_1$  and  $-1/k_2$ , then the closed-loop system is BIBO stable.

## The other cases

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### **$G$ : stable system**

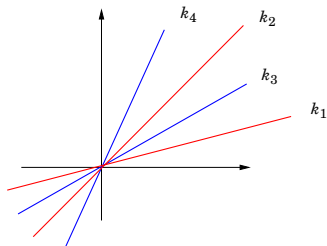
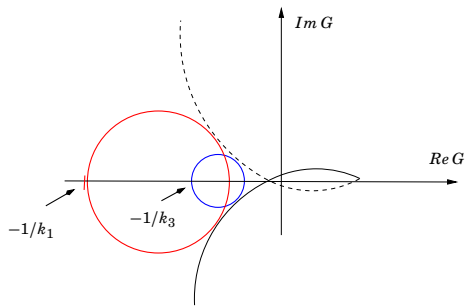
- ▶  $0 < k_1 < k_2$ : Stay outside circle
- ▶  $0 = k_1 < k_2$ : Stay to the right of the line  $\operatorname{Re} s = -1/k_2$
- ▶  $k_1 < 0 < k_2$ : Stay inside the circle

Other cases: Multiply  $f$  and  $G$  with  $-1$ .

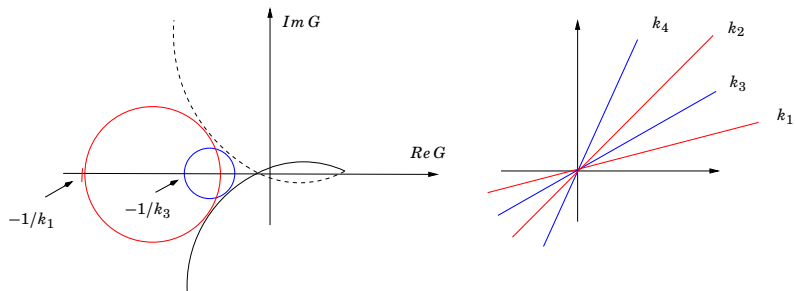
# Circle criterion / Sector conditions

What does it mean that we can get different sectors when using the circle criterion for a nonlinearity in feedback with a (fixed) linear system?

Can I have many different sector conditions, and what does that mean?

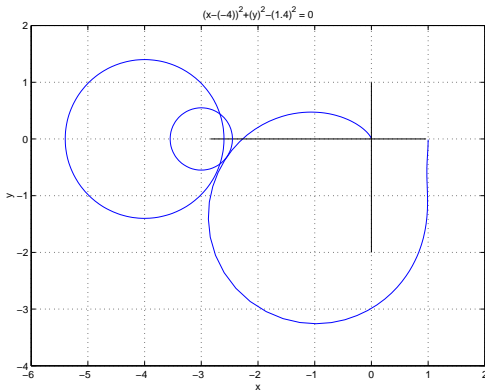




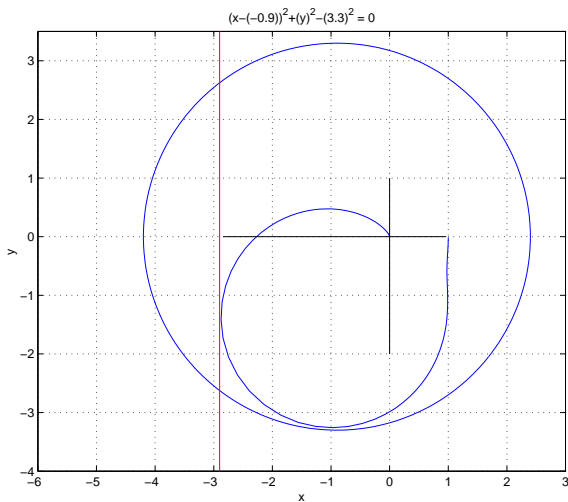


In the example above, the circle criterion can guarantee absolute stability for a nonlinearity which is bounded to **either** the sector  $[k_1, k_2]$  **or**  $[k_3, k_4]$  or in many other sectors, but NOT for a nonlinearity which is allowed to have a full variation within the sector  $[k_1, k_4]$ .

Example:  $G(s) = \frac{1000}{(s + 10)(s^2 + 2s + 100)}$   
 in negative feedback with a sector bounded nonlinearity.



sector  $[0.28, 0.41]$  or  $[0.19, 0.38]$



sector  $[0, 1/2.9]$  or

$[-0.41, 0.23]$

Small gain theorem will give symmetric sector  $[-0.27, 0.27]$  as  $\|G\| = 3.69$ .

# Questions

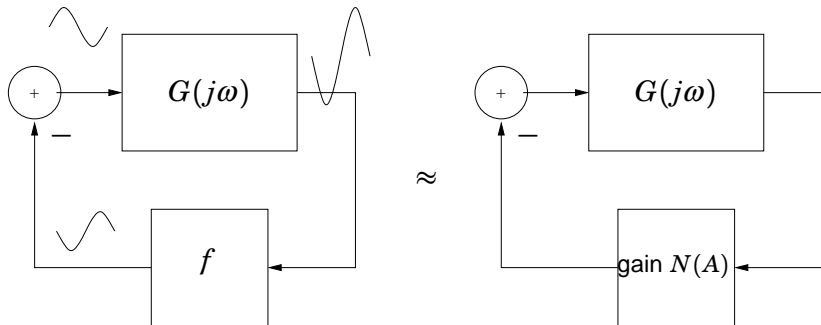
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Is it possible to draw phase portraits for systems of higher order than two?

Can the describing function method be improved by including more coefficients from the Fourier series expansion?

Are there criteria to verify the low-pass character needed in a describing function argument?

# Idea of Describing Function




Find frequency  $\omega$  and amplitude  $A$  such that

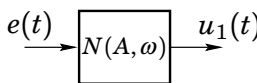
$$G(j\omega) \cdot N(A) = -1$$

# Idea of Describing Function

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$$e(t) = A \sin \omega t = \text{Im} (Ae^{i\omega t})$$

  $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$

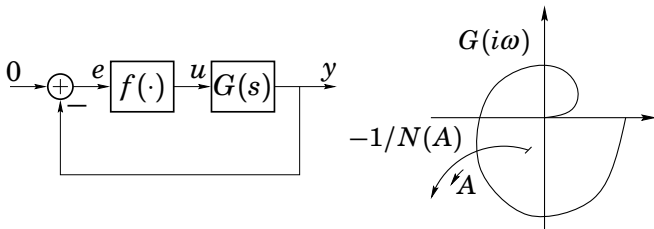
  $u_1(t) = a_1 \cos(\omega t) + b_1 \sin(\omega t)$   
 $= \text{Im} (N(A, \omega)Ae^{i\omega t})$

where the **describing function** is defined as

$$N(A, \omega) = \frac{b_1(\omega) + ia_1(\omega)}{A} \implies U(i\omega) \approx N(A, \omega)E(i\omega)$$

# Existence of Limit Cycles

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$$y = G(i\omega)u = -G(i\omega)N(A)y \quad \Rightarrow \quad G(i\omega) = -\frac{1}{N(A)}$$

The intersections of  $G(i\omega)$  and  $-1/N(A)$  give  $\omega$  and  $A$  for *possible* limit cycles.

Harder if  $N$  is a function of both  $A$  and  $\omega$ .

## Example from exam 20090601 (a)



Which one of the three describing functions below corresponds to the nonlinearity  $f(x)$  above?

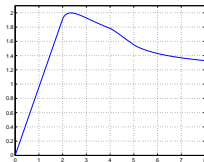
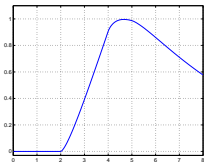
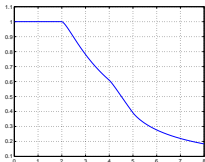
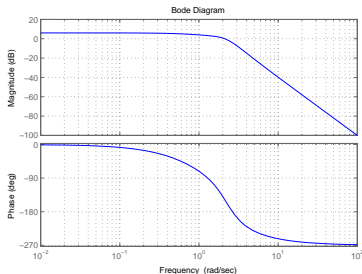
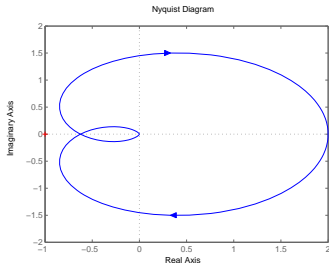


Figure: Describing functions 1 — 3



## Example from exam 20090601 (b)

Below we have the Nyquist and Bode curves of a linear system  $G$ . Assume that there exists non-linearities corresponding to the three describing functions on previous page, and that each of these would be used in a negative feedback connection with  $G$ . For which do we possibly get limit cycles? If so, state possible amplitudes of the limit cycles and if they are stable or unstable?

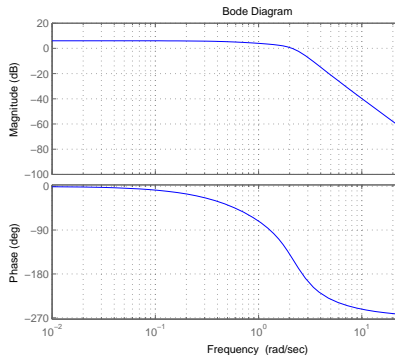
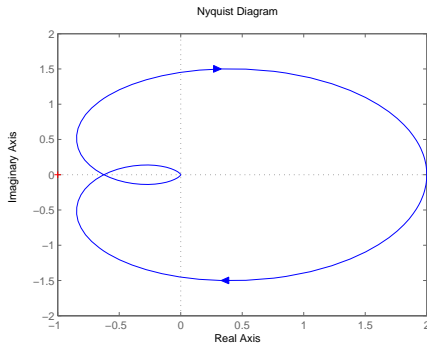


Since the third describing function fulfills that  $-\frac{1}{N(2)} = -\frac{1}{2}$  and  $G(i\omega_o) \approx -0.6$ , we understand that we have **two** intersections. The first intersection occurs when  $A \approx 1.8$  and the second intersection occurs when  $A \approx 4.5$ .

Examining the describing function around the first intersection, we see that  $-\frac{1}{N(A)}$  goes from the outside of  $G(i\omega)$  to the inside, with increasing  $A$ . Hence, we conclude that the possible limit cycle at  $A \approx 1.8$  is unstable. By similar argument, we understand that the possible limit cycle at  $A \approx 4.5$  is stable.

What would the corresponding frequency of the limit cycles in **(b)** be?

The frequency of all possible limit cycles is approximately 2.5 rad/s. To understand this, we see in the Bode plot that for  $\omega \approx 2.5$  we have that  $\arg(G(i\omega)) \approx -180$ .



## Question

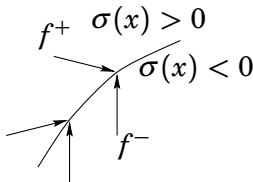
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Please repeat the most important facts about sliding modes.

# Sliding Modes

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$$\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}$$



The **sliding set** is where  $\sigma(x) = 0$  and  $f^+$  and  $f^-$  point towards  $\sigma(x) = 0$ .

The sliding dynamics are  $\dot{x} = \alpha f^+ + (1 - \alpha)f^-$ , where  $\alpha$  is obtained from  $\dot{\sigma} = \frac{\partial \sigma}{\partial x} \dot{x} = 0$ .

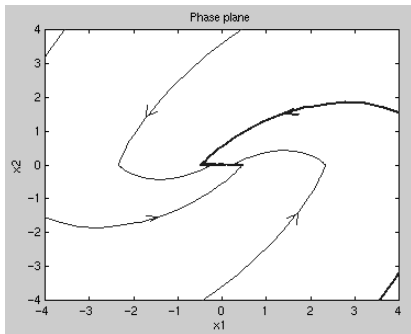
(More precisely, find  $\alpha$  such that the components of  $f^+$  and  $f^-$  perpendicular to the switching surface cancel.)

# Sliding Mode Dynamics

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The dynamics along the sliding surface  $\sigma(x) = 0$  can also be obtained by setting  $u = u_{eq} \in [-1, 1]$  such that  $\dot{\sigma}(x) = 0$ .

$u_{eq}$  is called the **equivalent control**.



Phase plane for example in lecture 12.

## Example

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$$\begin{aligned}\dot{x}_1 &= 1 - u/4 \\ \dot{x}_2 &= u, \\ u &= -\text{sign } x_2, \quad (\text{i.e., } \sigma(x) = x_2)\end{aligned}\tag{1}$$

What is the *sliding set* and what is *the sliding dynamics* for the system above?

If

$$\sigma(x) > 0 \Rightarrow u = -1 \Rightarrow f^+ = \begin{bmatrix} 5/4 \\ -1 \end{bmatrix}$$

$$\sigma(x) < 0 \Rightarrow u = +1 \Rightarrow f^- = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$$

## The sliding set:

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Find those values of the states at the switching curve for which

$$\nabla\sigma \cdot f^+ < 0$$

and

$$\nabla\sigma \cdot f^- > 0$$

(means that the vector fields on either side of  $\sigma(x)$  points towards  $\sigma(x)$ , i.e., the normal projection of  $f^+$  is negative and the normal projection of  $f^-$  on  $\sigma(x)$  is positive). If these conditions are not fulfilled we will just “flow through  $\sigma(x)$ ...”

In this example all the values along  $x_2 = 0$  will belong to the switching set. ( Compare with example from lecture 9 where the switching set will be restricted to  $x_2 = 0$  and  $-1 \leq x_1 \leq 1$ , see figure on slide above).



## The sliding dynamics:

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Alternative 1.a.: Solve via normal projection on  $\sigma$ :

Pick  $\alpha$  such that for  $\dot{x} = \alpha f^+ + (1 - \alpha) f^-$ , we get

$$\dot{\sigma} = 0 \Rightarrow \dot{x}_2 = \alpha f_2^+ + (1 - \alpha) f_2^- = 0$$

Alternative 1.b.: (same thing put in other words)

(The normal component on either side of the switch curve should balance out each other).

$$f_n^+ = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, f_n^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\alpha f_n^+ + (1 - \alpha) f_n^- = 0$  gives  $\alpha = 1/2$ ,

hence  $\dot{x} = \alpha f^+ + (1 - \alpha) f^-$  and  $\dot{x}_1 = 1$  is the sliding dynamics.

Alternative 2: Solve via Equivalent control

$$\dot{\sigma}(x)_{u=u_{eq}} = 0 \text{ and } \dot{\sigma} = \dot{x}_2 = u \Rightarrow u_{eq} = 0.$$

Hence  $\dot{x}_1 = 1 - u_{eq}/4 = 1$  is the sliding dynamics.

## Question

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Please repeat optimal control with some additional example

# Problem Formulation (1)

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$$\text{Minimize } \int_0^{t_f} L(x(t), u(t)) dt + \phi(x(t_f))$$

$$\dot{x}(t) = f(x(t), u(t))$$

$$u(t) \in U, \quad 0 \leq t \leq t_f, \quad t_f \text{ given}$$

$$x(0) = x_0$$

$$x(t) \in R^n, u(t) \in R^m$$

$U$  control constraints

## The Maximum Principle (18.2)

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Introduce the **Hamiltonian**

$$H(x, u, \lambda) = L(x, u) + \underbrace{\lambda^T(t)}_{1 \times n} \underbrace{f(x, u)}_{n \times 1}.$$

Suppose optimization problem (1) has a solution  $u^*(t), x^*(t)$ .  
Then the optimal solution must satisfy

$$\min_{u \in U} H(x^*(t), u, \lambda(t)) = H(x^*(t), u^*(t), \lambda(t)), \quad 0 \leq t \leq t_f,$$

where  $\lambda(t)$  solves the **adjoint equation**

$$\dot{\lambda}(t) = -H_x^T(x^*(t), u^*(t), \lambda(t)), \quad \lambda(t_f) = \phi_x^T(x^*(t_f))$$

where  $H_x = \frac{\partial H}{\partial x} = [\frac{\partial H}{\partial x_1} \dots \frac{\partial H}{\partial x_n}]$ ,  $\phi_x = \frac{\partial \phi}{\partial x}$ .

## Problem Formulation (2)

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As in (1) but with additions:

- ▶  $r$  end constraints

$$\Psi(x(t_f)) = \begin{pmatrix} \Psi_1(x(t_f)) \\ \vdots \\ \Psi_r(x(t_f)) \end{pmatrix} = 0$$

- ▶ free end time  $t_f$

## The Maximum Principle—General Case (18.4)

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Introduce the Hamiltonian

$$H(x, u, \lambda, n_0) = n_0 L(x, u) + \lambda^T(t) f(x, u)$$

Suppose optimization problem (2) has a solution  $u^*(t), x^*(t)$ . Then there is a vector function  $\lambda(t)$ , a number  $n_0 \geq 0$ , and a vector  $\mu \in R^r$  so that  $[n_0 \ \mu^T] \neq 0$  and

$$\min_{u \in U} H(x^*(t), u, \lambda(t), n_0) = H(x^*(t), u^*(t), \lambda(t), n_0), \quad 0 \leq t \leq t_f,$$

where

$$\begin{aligned}\dot{\lambda}(t) &= -H_x^T(x^*(t), u^*(t), \lambda(t), n_0) \\ \lambda(t_f) &= n_0 \phi_x^T(x^*(t_f)) + \Psi_x^T(x^*(t_f)) \mu\end{aligned}$$

## Free end time $t_f$

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If the choice of  $t_f$  is included in the optimization then two cases:  $n_0 = 1$  or  $n_0 = 0$ .

Also, there is an extra constraint:

$$H(x^*(t_f), u^*(t_f), \lambda(t_f), n_0) = 0$$

## Example: Optimal storage control

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$$\begin{aligned} &\text{Minimize } \int_0^{t_f} [u(t)e^{rt} + cx(t)]dt \\ &\text{subject to } \begin{cases} \dot{x} = u & 0 \leq u \leq M \\ x(0) = 0 \\ x(t_f) \geq A \end{cases} \end{aligned}$$

$x$  = stock size

$u$  = production rate

$r$  = production cost growth rate

$c$  = storage cost



## Optimal storage control II

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Hamiltonian

$$H = L(x, u) + \lambda(t)^T f(x, u) = ue^{rt} + cx + \lambda u$$

Adjoint equation

$$\begin{aligned}\dot{\lambda} &= -\frac{\partial H}{\partial x} = -c \\ \lambda &= b - ct\end{aligned}$$

At optimality

$$\begin{aligned}H(x^*, u^*, \lambda) &= \min_u H(x^*, u, \lambda) = cx + \min_u [u(e^{rt} + b - ct)] \\ u^* &= \begin{cases} M & \text{if } e^{rt} + b - ct < 0 \\ 0 & \text{if } e^{rt} + b - ct > 0 \end{cases}\end{aligned}$$

## Optimal storage control III

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$$u^*(t) = \begin{cases} M & \text{if } e^{rt} + b - ct < 0 \\ 0 & \text{if } e^{rt} + b - ct > 0 \end{cases} = \begin{cases} 0 & 0 \leq t \leq t_1 \\ M & t_1 \leq t \leq t_2 \\ 0 & t_2 \leq t \leq t_f \end{cases}$$

The condition  $x(t_f) \geq A$  gives that  $M(t_2 - t_1) = A$ .

What remains is the scalar optimization problem

$$\min_{0 \leq s \leq A/M} \left[ \int_s^{s+A/M} M(e^{rt} + ct) dt + \int_{s+A/M}^{t_f} cA dt \right]$$