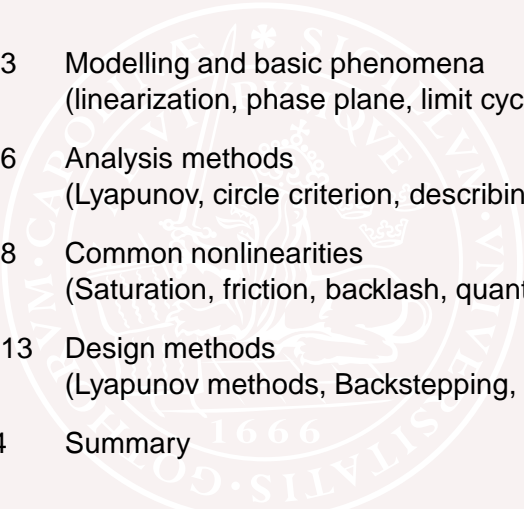


# Lecture 9 — Nonlinear Control Design

- Exact-linearization
- Lyapunov-based design
  - Lab 2
  - Adaptive control
  - Backstepping
- Hybrid / Piece-wise linear control
  - NOTE: Only overview!

Literature: [Khalil, ch.s 13, 14.2,14.3] and [Glad-Ljung,ch.17]

# Course Outline

- 
- Lecture 1-3    Modelling and basic phenomena  
(linearization, phase plane, limit cycles)
- Lecture 4-6    Analysis methods  
(Lyapunov, circle criterion, describing functions)
- Lecture 7-8    Common nonlinearities  
(Saturation, friction, backlash, quantization)
- Lecture 9-13   Design methods  
(Lyapunov methods, Backstepping, Optimal control)
- Lecture 14    Summary

# Exact Feedback Linearization

## Idea:

Find state feedback  $u = u(x, v)$  so that the nonlinear system

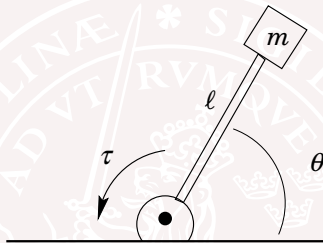
$$\dot{x} = f(x) + g(x)u$$

turns into the linear system

$$\dot{x} = Ax + Bv$$

and then apply linear control design method.

# Exact linearization: example [one-link robot]



$$m\ell^2\ddot{\theta} + d\dot{\theta} + m\ell g \cos \theta = u$$

where  $d$  is the viscous damping.

The control  $u = \tau$  is the applied torque

Design state feedback controller  $u = u(x)$  with  $x = (\theta, \dot{\theta})^T$

Introduce new control variable  $v$  and let

$$u = m\ell^2 v + d\dot{\theta} + m\ell g \cos \theta$$

Then

$$\ddot{\theta} = v$$

Choose e.g. a PD-controller

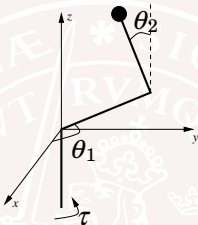
$$v = v(\theta, \dot{\theta}) = k_p(\theta_{\text{ref}} - \theta) - k_d\dot{\theta}$$

This gives the closed-loop system:

$$\ddot{\theta} + k_d\dot{\theta} + k_p\theta = k_p\theta_{\text{ref}}$$

Hence,  $u = m\ell^2[k_p(\theta - \theta_{\text{ref}}) - k_d\dot{\theta}] + d\dot{\theta} + m\ell g \cos \theta$

# Multi-link robot (n-joints)



General form

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = u, \quad \theta \in R^n$$

Called *fully* actuated if  $n$  indep. actuators,

$M$   $n \times n$  inertia matrix,  $M = M^T > 0$

$C\dot{\theta}$   $n \times 1$  vector of centrifugal and Coriolis forces

$G$   $n \times 1$  vector of gravitation terms

# Computed torque

The computed torque  
(also known as "Exact linearization", "dynamic inversion" , etc. )

$$\begin{aligned}u &= M(\theta)v + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) \\v &= K_p(\theta_{ref} - \theta) - K_d\dot{\theta},\end{aligned}\tag{1}$$

gives closed-loop system

$$\ddot{\theta} + K_d\dot{\theta} + K_p\theta = K_p\theta_{Ref}$$

The matrices  $K_d$  and  $K_p$  can be chosen diagonal (no cross-terms) and then this decouples into  $n$  independent second-order equations.

# Lyapunov-Based Control Design Methods

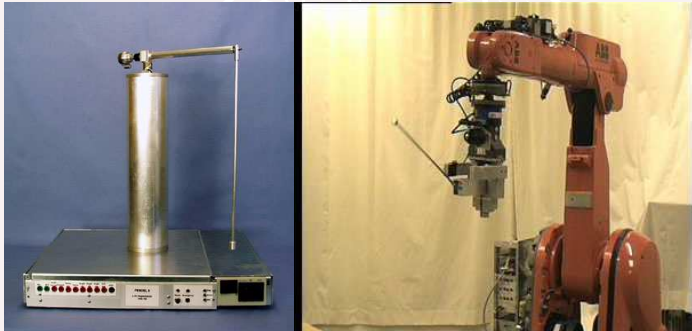
$$\dot{x} = f(x, u)$$

- Select Lyapunov function  $V(x)$  for stability verification
- Find state feedback  $u = u(x)$  that makes  $V$  decreasing
- Method depends on structure of  $f$

Examples are energy shaping as in Lab 2 and, e.g., **Back-stepping control design**, which require certain  $f$  discussed later.



# Lab 2 : Energy shaping for swing-up control



Use Lyapunov-based design for swing-up control.

## Lab 2 : Energy shaping for swing-up control



Rough outline of method to get the pendulum to the upright position

- Find expression for total energy  $E$  of the pendulum (potential energy + kinetic energy)
- Let  $E_n$  be energy in upright position.
- Look at deviation  $V = \frac{1}{2}(E - E_n)^2 \geq 0$
- Find "swing strategy" of control torque  $u$  such that  $\dot{V} \leq 0$

# Example of Lyapunov-based design

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 2x_1x_2^2 + u \\ \dot{x}_2 &= -x_2^3 - x_2,\end{aligned}\tag{2}$$

Find a nonlinear feedback control law which makes the origin globally asymptotically stable.

We try the standard Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2} (x_1^2 + x_2^2),$$

which is radially unbounded,  $V(0, 0) = 0$ , and  $V(x_1, x_2) > 0 \ \forall (x_1, x_2) \neq (0, 0)$ .

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## Example - cont'd

$$\begin{aligned}\dot{V} &= \dot{x}_1 x_1 + \dot{x}_2 x_2 = (-3x_1 + 2x_1 x_2^2 + u)x_1 + (-x_2^3 - x_2)x_2 \\ &= -3x_1^2 - x_2^2 + ux_1 + 2x_1^2 x_2^2 - x_2^4\end{aligned}$$

We would like to have

$$\dot{V} < 0 \quad \forall (x_1, x_2) \neq (0, 0)$$

Inserting the control law,  $u = -2x_1 x_2^2$ , we get

$$\dot{V} = -3x_1^2 - x_2^2 - \underbrace{2x_1^2 x_2^2 + 2x_1^2 x_2^2}_{=0} - x_2^4 = -3x_1^2 - x_2^2 - x_2^4 < 0, \quad \forall x \neq 0$$

## Example - cont'd

$$\begin{aligned}\dot{V} &= \dot{x}_1 x_1 + \dot{x}_2 x_2 = (-3x_1 + 2x_1 x_2^2 + u)x_1 + (-x_2^3 - x_2)x_2 \\ &= -3x_1^2 - x_2^2 + ux_1 + 2x_1^2 x_2^2 - x_2^4\end{aligned}$$

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Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u\end{aligned}\tag{3}$$

Find a globally asymptotically stabilizing control law  $u = u(x)$ .

Attempt 1: Try the standard Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2} (x_1^2 + x_2^2),$$

which is radially unbounded,  $V(0, 0) = 0$ , and  $V(x_1, x_2) > 0 \forall (x_1, x_2) \neq (0, 0)$ .

$$\dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2 = x_2^3 \cdot x_1 + u \cdot x_2 = x_2 \underbrace{(x_2^2 x_1 + u)}_{-x_2} = -x_2^2 \leq 0$$

where we chose

$$u = -x_2 - x_2^2 x_1$$

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u\end{aligned}\tag{3}$$

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where we chose

$$u = -x_2 - x_2^2 x_1$$



However  $\dot{V} = 0$  as soon as  $x_2 = 0$  (Note:  $x_1$  could be anything).

According to LaSalle's theorem the set

$$E = \{x | \dot{V} = 0\} = \{(x_1, 0)\} \forall x_1$$

What is the largest invariant subset  $M \subseteq E$ ?

Plugging in the control law  $u = -x_2 - x_2^2 x_1$ , we get

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_2 - x_2^2 x_1\end{aligned}\tag{4}$$

Observe that if we start anywhere on the line  $\{(x_1, 0)\}$  we will stay in the same point as both  $\dot{x}_1 = 0$  and  $\dot{x}_2 = 0$ , thus  $M=E$  and we will not converge to the origin, but get stuck on the line  $x_2 = 0$ .

Draw phase-plot with e.g., pplane and study the behaviour.

## Attempt 2:

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u\end{aligned}\tag{5}$$

Try the Lyapunov function **candidate**

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4,$$

which satisfies

- $V(0, 0) = 0$
- $V(x_1, x_2) > 0, \quad \forall (x_1, x_2) \neq (0, 0).$
- radially unbounded,
- compute

$$\dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2^3 = x_2^3(x_1 + u) = -x_2^4 \leq 0$$

↑

if we use  $u = -x_1 - x_2$

With

$$u = -x_1 - x_2$$

we get the dynamics

$$\begin{aligned}\dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_1 - x_2\end{aligned}\tag{6}$$

$\dot{V} = 0$  if  $x_2 = 0$ , thus

$$E = \{x | \dot{V} = 0\} = \{(x_1, 0) \forall x_1\}$$

However, now the only possibility to stay on  $x_2 = 0$  is if  $x_1 = 0$ , (else  $\dot{x}_2 \neq 0$  and we will leave the line  $x_2 = 0$ ).

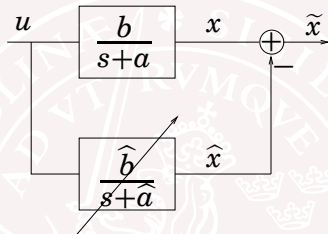
Thus, the largest invariant set

$$M = (0, 0)$$

According to the Invariant Set Theorem (LaSalle) all solutions will end up in  $M$  and so the origin is GAS.

Draw phase-plot with e.g., pplane and study the behaviour.

# Adaptive Noise Cancellation Revisited



$$\dot{x} + ax = bu$$

$$\dot{\hat{x}} + \hat{a}\hat{x} = \hat{b}u$$

Introduce  $\tilde{x} = x - \hat{x}$ ,  $\tilde{a} = a - \hat{a}$ ,  $\tilde{b} = b - \hat{b}$ .

Want to design adaptation law so that  $\tilde{x} \rightarrow 0$

Let us try the Lyapunov function

$$\begin{aligned} V &= \frac{1}{2}(\tilde{x}^2 + \gamma_a \tilde{a}^2 + \gamma_b \tilde{b}^2) \\ \dot{V} &= \tilde{x}\dot{\tilde{x}} + \gamma_a \tilde{a}\dot{\tilde{a}} + \gamma_b \tilde{b}\dot{\tilde{b}} = \\ &= \tilde{x}(-a\tilde{x} - \tilde{a}\tilde{x} + \tilde{b}u) + \gamma_a \tilde{a}\dot{\tilde{a}} + \gamma_b \tilde{b}\dot{\tilde{b}} = -a\tilde{x}^2 \end{aligned}$$

where the last equality follows if we choose

$$\dot{\tilde{a}} = -\hat{a} = \frac{1}{\gamma_a} \tilde{x}\hat{x} \quad \dot{\tilde{b}} = -\hat{b} = -\frac{1}{\gamma_b} \tilde{x}u$$

Invariant set:  $\tilde{x} = 0$ .

This proves that  $\tilde{x} \rightarrow 0$ .

(The parameters  $\tilde{a}$  and  $\tilde{b}$  do not necessarily converge:  $u \equiv 0$ .)

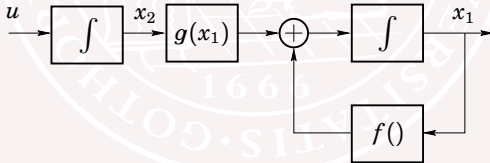
# Back-Stepping Control Design

We want to design a state feedback  $u = u(x)$  that stabilizes

$$\begin{aligned}\dot{x}_1 &= f(x_1) + g(x_1)x_2 \\ \dot{x}_2 &= u\end{aligned}\tag{7}$$

at  $x = 0$  with  $f(0) = 0$ .

**Idea:** See the system as a cascade connection. Design controller first for the inner loop and then for the outer.



Suppose the partial system

$$\dot{x}_1 = f(x_1) + g(x_1)\bar{v}$$

can be stabilized by  $\bar{v} = \phi(x_1)$  and there exists Lyapunov fcn  $V_1 = V_1(x_1)$  such that

$$\dot{V}_1(x_1) = \frac{dV_1}{dx_1} \left( f(x_1) + g(x_1)\phi(x_1) \right) \leq -W(x_1)$$

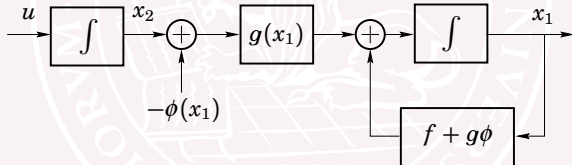
for some positive definite function  $W$ .

# The Trick

Equation (7) can be rewritten as

$$\dot{x}_1 = f(x_1) + g(x_1)\phi(x_1) + g(x_1)[x_2 - \phi(x_1)]$$

$$\dot{x}_2 = u$$

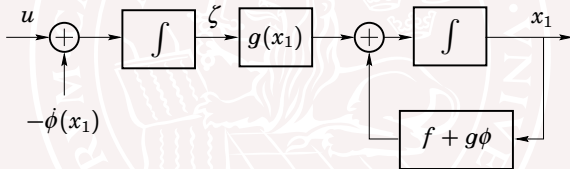




Introduce new state  $\zeta = x_2 - \phi(x_1)$  and control  $v = u - \dot{\phi}$ :

$$\dot{x}_1 = f(x_1) + g(x_1)\phi(x_1) + g(x_1)\zeta$$

$$\dot{\zeta} = v$$



Consider  $V_2(x_1, x_2) = V_1(x_1) + \zeta^2/2$ . Then,

$$\begin{aligned}\dot{V}_2(x_1, x_2) &= \frac{dV_1}{dx_1} \left( f(x_1) + g(x_1)\phi(x_1) \right) + \frac{dV_1}{dx_1} g(x_1)\zeta + \zeta v \\ &\leq -W(x_1) + \frac{dV_1}{dx_1} g(x_1)\zeta + \zeta v\end{aligned}$$

Choosing

$$v = -\frac{dV_1}{dx_1} g(x_1) - k\zeta, \quad k > 0$$

gives

$$\dot{V}_2(x_1, x_2) \leq -W(x_1) - k\zeta^2$$

Hence,  $x = 0$  is asymptotically stable for (7) with control law  $u(x) = \dot{\phi}(x) + v(x)$ .

If  $V_1$  radially unbounded, then global stability.

# Back-Stepping Lemma

**Lemma:** Let  $z = (x_1, \dots, x_{k-1})^T$  and

$$\dot{z} = f(z) + g(z)x_k$$

$$\dot{x}_k = u$$

Assume  $\phi(0) = 0$ ,  $f(0) = 0$ ,

$$\dot{z} = f(z) + g(z)\phi(z)$$

stable, and  $V(z)$  a Lyapunov fcn (with  $\dot{V} \leq -W$ ). Then,

$$u = \frac{d\phi}{dz} \left( f(z) + g(z)x_k \right) - \frac{dV}{dz} g(z) - (x_k - \phi(z))$$

stabilizes  $x = 0$  with  $V(z) + (x_k - \phi(z))^2/2$  being a Lyapunov fcn.

# Strict Feedback Systems

Back-stepping Lemma can be applied to stabilize systems on strict feedback form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n) + g_n(x_1, \dots, x_n)u\end{aligned}$$

where  $g_k \neq 0$

**Note:**  $x_1, \dots, x_k$  do not depend on  $x_{k+2}, \dots, x_n$ .

# Back-Stepping

Back-Stepping Lemma can be applied **recursively** to a system

$$\dot{x} = f(x) + g(x)u$$

on strict feedback form.

Back-stepping generates stabilizing feedbacks  $\phi_k(x_1, \dots, x_k)$  (equal to  $u$  in Back-Stepping Lemma) and Lyapunov functions

$$V_k(x_1, \dots, x_k) = V_{k-1}(x_1, \dots, x_{k-1}) + [x_k - \phi_{k-1}]^2/2$$

by “stepping back” from  $x_1$  to  $u$

Back-stepping results in the final state feedback

$$u = \phi_n(x_1, \dots, x_n)$$

# Example

Design back-stepping controller for

$$\dot{x}_1 = x_1^2 + x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

**Step 0** Verify strict feedback form

**Step 1** Consider first subsystem

$$\dot{x}_1 = x_1^2 + \phi_1(x_1), \quad \dot{x}_2 = u_1$$

where  $\phi_1(x_1) = -x_1^2 - x_1$  stabilizes the first equation. With  $V_1(x_1) = x_1^2/2$ , Back-Stepping Lemma gives

$$u_1 = (-2x_1 - 1)(x_1^2 + x_2) - x_1 - (x_2 + x_1^2 + x_1) = \phi_2(x_1, x_2)$$

$$V_2 = x_1^2/2 + (x_2 + x_1^2 + x_1)^2/2$$

## Step 2 Applying Back-Stepping Lemma on

$$\dot{x}_1 = x_1^2 + x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u$$

gives

$$\begin{aligned} u = u_2 &= \frac{d\phi_2}{dz} \left( f(z) + g(z)x_n \right) - \frac{dV_2}{dz} g(z) - (x_n - \phi_2(z)) \\ &= \frac{\partial \phi_2}{\partial x_1} (x_1^2 + x_2) + \frac{\partial \phi_2}{\partial x_2} x_3 - \frac{\partial V_2}{\partial x_2} - (x_3 - \phi_2(x_1, x_2)) \end{aligned}$$

which globally stabilizes the system.

# Hybrid Control

Control problems where there is a mixture between continuous states and discrete state variables.

Continuous states: position, velocity, temperature, pressure

Discrete states: on/off variables, controller modes, loss of actuators, loss of sensors, relays, etc

Discontinuous differential equations

Much active field, much left to understand

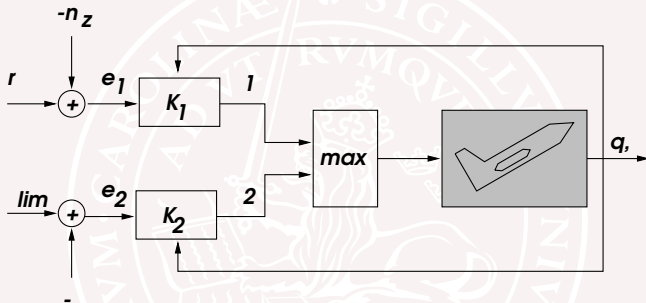


# Example of hybrid control

Control law that switches between different modes, e.g. between

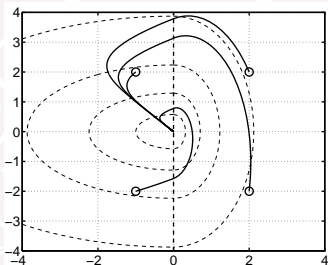
- Time optimal control – during large set point changes
- Linear control – close to set point

# Aircraft Example



(Branicky, 1993)

# Phase Plane



No common *quadratic* Lyapunov function exists.

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$

# Piecewise quadratic Lyapunov functions

$$V(x) = \begin{cases} x^* P x & \text{if } x_1 < 0 \\ x^* P x + \eta x_1^2 & \text{if } x_1 \geq 0 \end{cases}$$

The matrix inequalities

$$A_1^* P + P A_1 < 0$$

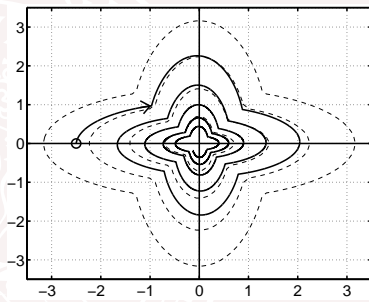
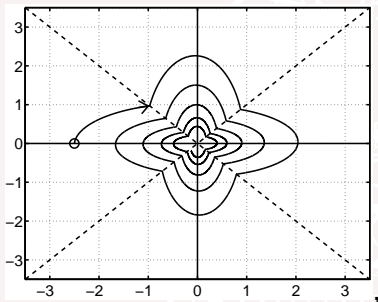
$$P > 0$$

$$A_2^* (P + \eta E^* E) + (P + \eta E^* E) A_2 < 0$$

$$P + \eta E^* E > 0$$

with  $E = [1 \ 0]$ , have the solution  $P = \text{diag}\{1, 3\}$ ,  $\eta = 7$ .

# Flower Example



# Next Lecture

- Optimization.

Read chapter 18 in [Glad & Ljung] for preparation.

