

L1-L6 Analysis Methods (Linearization, Lyapunov, Circle criterion, Describing functions)

L7,L9 Special nonlinearities

L10-L11 Optimal control

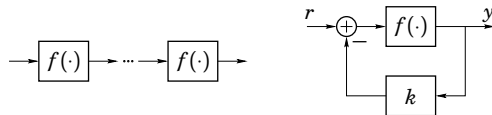
L8,L12,L13 Synthesis of nonlinear controllers

L14 Course summary

History of the Feedback Amplifier

New York–San Francisco communication link 1914.

High signal amplification with low distortion was needed.



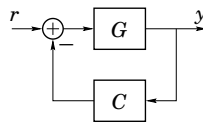
Feedback amplifiers was the solution!

Black, Bode, and Nyquist at Bell Labs 1920–1950.

The Sensitivity Function $S = (1 + GC)^{-1}$

The closed-loop system is

$$G_{cl} = \frac{G}{1 + GC}$$



Small perturbations dG in G gives

$$dG_{cl} = \frac{dG}{1 + GC} - \frac{GCdG}{(1 + GC)^2} = \frac{dG}{(1 + GC)^2}$$

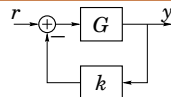
so that

$$\frac{dG_{cl}}{G_{cl}} = \underbrace{\frac{1}{1 + GC}}_S \frac{dG}{G}$$

S is the closed-loop **sensitivity** to open-loop perturbations.

Example—Distortion Reduction

Let $G = 1000$ with distortion $dG/G = 0.1$.



Choose $k = 0.1$ so that $S = (1 + Gk)^{-1} \approx 0.01$. Then

$$\frac{dG_{cl}}{G_{cl}} = S \frac{dG}{G} \approx 0.001$$

One hundred feedback amplifiers give total amplification

$$G_{tot} = (G_{cl})^{100} \approx 10^{100}$$

and total distortion

$$\frac{dG_{tot}}{G_{tot}} = (1 + 10^{-3})^{100} - 1 \approx 0.1$$

Today's Goal You should be able to analyze and design

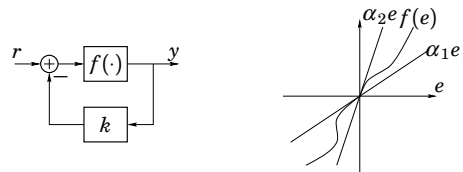
- ▶ high-gain control system
- ▶ sliding mode controller

Material

- ▶ Lecture notes

- ▶ Khalil: Sec 14.1.1 (pp.552–563)
- ▶ Slotine and Li: Section 7.1, 8.5
- ▶ Chapter 10 in *Adaptive Control* by Åström & Wittenmark

Linearization Through High Gain



$$\alpha_1 e \leq f(e) \leq \alpha_2 e \quad \text{gives} \quad \frac{\alpha_1}{1 + \alpha_1 k} r \leq y \leq \frac{\alpha_2}{1 + \alpha_2 k} r$$

so if $\alpha_1 k, \alpha_2 k \gg 1$ then

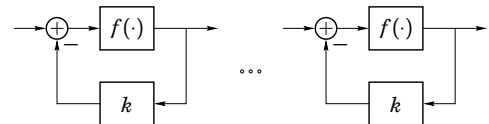
$$y \approx \frac{1}{k} r$$

regardless of the nonlinearity. (Easier to design $k < 1$ with high accuracy.)

Distortion Reduction via Feedback Amplifiers

The feedback reduces distortion in each link.

Several links give distortion-free high gain.

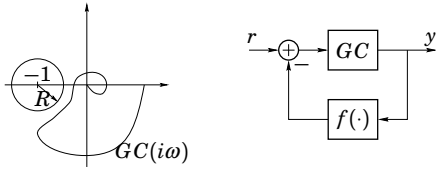


Transcontinental Communication Revolution

The feedback amplifier was patented by Black 1937.

Year	Channels	Loss (dB)	No amp's
1914	1	60	3–6
1923	1–4	150–400	6–20
1938	16	1000	40
1941	480	30000	600

Sensitivity and the Circle Criterion



$|S(i\omega)| = 1/R$ corresponds in the Nyquist diagram to a circle with center in -1 and radius R .

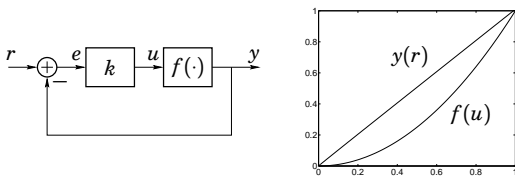
Circle criterion gives stability if

$$\frac{1}{1+R} \leq \frac{f(y)}{y} \leq \frac{1}{1-R}$$

$|S(i\omega)|$ small implies low sensitivity to nonlinearities.

High Gain Linearization of Static Nonlinearity

Same idea often applicable

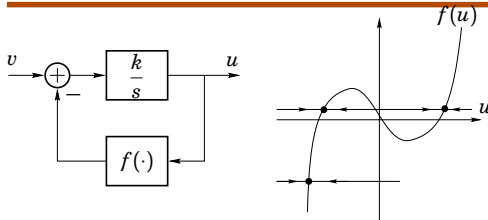


Linearization of $f(u) = u^2$ through feedback.

The case $k = 100$ is shown in the plot: $y(r) \approx r$.

Warning: High gain can give a noise sensitive system.

How to Obtain f^{-1} from f



$$\dot{u} = k(v - f(u))$$

If $k > 0$ large and $df/du > 0$, then $\dot{u} \rightarrow 0$ and

$$0 = k(v - f(u)) \quad \text{that is} \quad u = f^{-1}(v)$$

Note that the function f above does not have a well-defined inverse! What "inverse" can you get with the scheme above?

A Control Design Idea, and a Problem

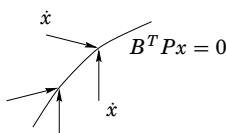
Assume $V(x) = x^T P x$, $P > 0$, represents the energy of

$$\dot{x} = Ax + Bu, \quad u \in [-1, 1]$$

Idea: Choose u such that V decays as fast as possible

$$\begin{aligned} \dot{V} &= x^T (A^T P + AP)x + 2B^T P x \cdot u \\ u &= -\text{sgn}(B^T P x) \end{aligned}$$

The following situation might then occur ("system is not Lipschitz")



Small Sensitivity Allows Large Uncertainty

If $|S(i\omega)|$ is small, we can choose R large (close to one).

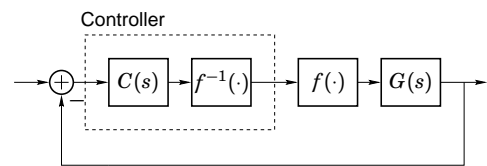
This corresponds to a large sector for $f(\cdot)$.

Hence, $|S(i\omega)|$ small implies low sensitivity to nonlinearities.



Inverting Nonlinearities

Compensation of static nonlinearity through inversion:



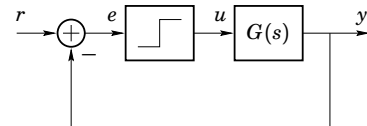
Should be combined with feedback as in the figure!

What if (a) $f(x) = x^3$? (b) $f(x) = x^2$?

On-Off Control

On-off control is the simplest control strategy.

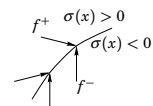
Common in temperature control, level control etc.



The relay feedback corresponds to extreme high-gain control.

Sliding Modes

$$\dot{x} = \begin{cases} f^+(x), & \sigma(x) > 0 \\ f^-(x), & \sigma(x) < 0 \end{cases}$$



The **switching set/ sliding set** is where $\sigma(x) = 0$ and f^+ and f^- point towards $\sigma(x) = 0$.

The **switching set/ sliding set** is given by x such that

$$\sigma(x) = 0$$

$$\frac{\partial \sigma}{\partial x} f^+ = (\nabla \sigma) f^+ < 0$$

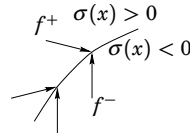
$$\frac{\partial \sigma}{\partial x} f^- = (\nabla \sigma) f^- > 0$$

Note: If f^+ and f^- point "in the same direction" on both sides of the set $\sigma(x) = 0$ then the solution curves will just pass through and this region will not belong to the sliding set.

Sliding Mode

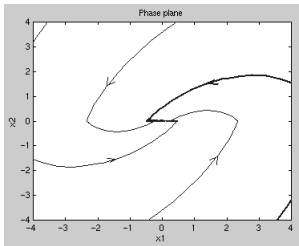
If f^+ and f^- both point towards $\sigma(x) = 0$, what will happen then?

The sliding dynamics are $\dot{x} = \alpha f^+ + (1 - \alpha) f^-$, where α is obtained from $\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial x} \cdot \dot{x} = 0$.



More precisely, find α such that the components of f^+ and f^- perpendicular to the switching surface cancel: $\alpha f^+_{\perp} + (1 - \alpha) f^-_{\perp} = 0$. The resulting dynamics is then the sum of the corresponding components along the surface.

Sliding Mode Dynamics



The dynamics along the sliding set in $\sigma(x) = 0$ can also be obtained by finding $u = u_{eq} \in [-1, 1]$ such that $\dot{\sigma}(x) = 0$.

u_{eq} is called the **equivalent control**.

Equivalent Control

Assume

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ u &= -\text{sgn } \sigma(x)\end{aligned}$$

has a sliding set on $\sigma(x) = 0$. Then, for $x(t)$ staying on the sliding set we should have

$$0 = \dot{\sigma}(x) = \frac{\partial \sigma}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial \sigma}{\partial x} (f(x) + g(x)u)$$

The equivalent control is thus given by

$$u_{eq} = -\left(\frac{\partial \sigma}{\partial x} g(x)\right)^{-1} \frac{\partial \sigma}{\partial x} f(x)$$

if the inverse exists.

More on the Sliding Dynamics

If $CB > 0$ then the dynamics along a sliding set in $Cx = 0$ is

$$\dot{x} = Ax + Bu_{eq} = \left(I - \frac{BC}{CB}\right)Ax,$$

One can show that the eigenvalues of $(I - BC/CB)A$ equals the zeros of $G(s) = C(sI - A)^{-1}B$. (exercise for PhD students)

4 minute exercise

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u = Ax + Bu \\ u &= -\text{sgn } \sigma(x) = -\text{sgn } x_2 = -\text{sgn}(Cx)\end{aligned}$$

which means that

$$\dot{x} = \begin{cases} Ax - B, & x_2 > 0 \\ Ax + B, & x_2 < 0 \end{cases}$$

Determine the *switching set* and the *sliding dynamics* on this set.

Example (cont'd)

Finding $u = u_{eq}$ such that $\dot{\sigma}(x) = \dot{x}_2 = 0$ gives

$$0 = \dot{x}_2 = x_1 - \underbrace{x_2}_{=0} + u_{eq} = x_1 + u_{eq}$$

Insert $u_{eq} = -x_1$ in the equation for \dot{x}_1 :

$$\dot{x}_1 = -\underbrace{x_2}_{=0} + u_{eq} = -x_1$$

gives the dynamics on the sliding set (where $x_2 = 0$)

Remember: $u_{eq} \in [-1, 1]$ so can only satisfy $u_{eq} = -x_1$ on the interval $x_1 \in [-1, 1]$!

Equivalent Control for Linear System

$$\begin{aligned}\dot{x} &= Ax + Bu \\ u &= -\text{sgn } \sigma(x) = -\text{sgn}(Cx)\end{aligned}$$

Assume $CB > 0$. The sliding set lies in $\sigma(x) = Cx = 0$.

$$0 = \dot{\sigma}(x) = \frac{d\sigma}{dx} (f(x) + g(x)u) = C(Ax + Bu_{eq})$$

gives $u_{eq} = -CAx/CB$.

Example (cont'd) For the previous system

$$u_{eq} = -CAx/CB = -\begin{pmatrix} 1 & -1 \end{pmatrix} x = -x_1 + x_2 = -x_1,$$

because $\sigma(x) = x_2 = 0$. Same result as above.

Design of Sliding Mode Controller

Idea: Design a control law that forces the state to $\sigma(x) = 0$. Choose $\sigma(x)$ such that the sliding mode tends to the origin.

Assume system has form

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x) + g_1(x)u \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = f(x) + g(x)u$$

Choose control law

$$u = -\frac{p^T f(x)}{p^T g(x)} - \frac{\mu}{p^T g(x)} \text{sgn } \sigma(x),$$

where $\mu > 0$ is a design parameter, $\sigma(x) = p^T x$, and $p^T = \begin{pmatrix} p_1 & \dots & p_n \end{pmatrix}$ represents a stable polynomial.

Sliding Mode Control gives Closed-Loop Stability

Consider $\mathcal{V}(x) = \sigma^2(x)/2$ with $\sigma(x) = p^T x$. Then,

$$\dot{\mathcal{V}} = \sigma(x)\dot{\sigma}(x) = x^T p(p^T f(x) + p^T g(x)u)$$

With the chosen control law, we get

$$\dot{\mathcal{V}} = -\mu\sigma(x) \operatorname{sgn} \sigma(x) \leq 0$$

so x tend to $\sigma(x) = 0$.

$$\begin{aligned} 0 = \sigma(x) &= p_1 x_1 + \dots + p_{n-1} x_{n-1} + p_n x_n \\ &= p_1 x_n^{(n-1)} + \dots + p_{n-1} x_n^{(1)} + p_n x_n^{(0)} \end{aligned}$$

where $x^{(k)}$ denote time derivative. P stable gives that $x(t) \rightarrow 0$.

Note: \mathcal{V} is itself not a true Lyapunov function. It only guarantees convergence to the line $\sigma(x) = p^T x$.

Example—Sliding Mode Controller

Design state-feedback controller for

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x \end{aligned}$$

Choose $p_1 s + p_2 = s + 1$ so that $\sigma(x) = x_1 + x_2$. The controller is given by

$$\begin{aligned} u &= -\frac{p^T A x}{p^T B} - \frac{\mu}{p^T B} \operatorname{sgn} \sigma(x) \\ &= -2x_1 - \mu \operatorname{sgn}(x_1 + x_2) \end{aligned}$$

Time Plots

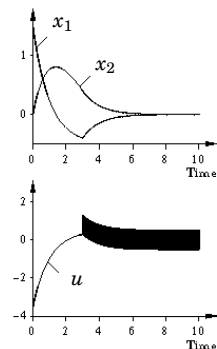
Initial condition
 $x(0) = \begin{pmatrix} 1.5 & 0 \end{pmatrix}^T$.

Simulation agrees well
with time to switch

$$t_s = \frac{\sigma_0}{\mu} = 3$$

and sliding dynamics

$$\dot{y} = -y$$



Time to Switch

Consider an initial point x such that $\sigma_0 = \sigma(x) > 0$. Then

$$\sigma(x)\dot{\sigma}(x) = -\mu\sigma(x) \operatorname{sgn} \sigma(x)$$

so

$$\dot{\sigma}(x) = -\mu$$

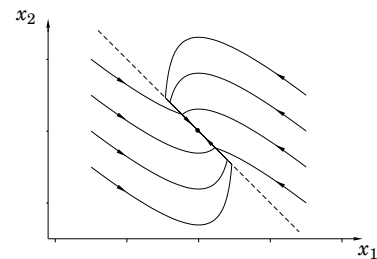
Hence, the time to the first switch is

$$t_s = \frac{\sigma_0}{\mu} < \infty$$

Note that $t_s \rightarrow 0$ as $\mu \rightarrow \infty$.

Phase Portrait

Simulation with $\mu = 0.5$. Note the sliding set is in $\sigma(x) = x_1 + x_2$.



The Sliding Mode Controller is Robust

Assume that only a model $\dot{x} = \hat{f}(x) + \hat{g}(x)u$ of the true system $\dot{x} = f(x) + g(x)u$ is known. Still, however,

$$\dot{\mathcal{V}} = \sigma(x) \left[\frac{p^T (f\hat{g}^T - \hat{f}g^T)p}{p^T \hat{g}} - \mu \frac{p^T g}{p^T \hat{g}} \operatorname{sgn} \sigma(x) \right] < 0$$

if $\operatorname{sgn}(p^T g) = \operatorname{sgn}(p^T \hat{g})$ and $\mu > 0$ is sufficiently large.

The closed-loop system is thus quite robust against model errors!

(High gain control with stable open loop zeros)

Implementation

A relay with hysteresis or a smooth (e.g. linear) region is often used in practice.

Choice of hysteresis or smoothing parameter can be critical for performance

More complicated structures with several relays possible. Harder to design and analyze.

Next Lectures

- L13: Other synthesis methods
- L14: Course summary