Lecture 2

- Linearization
- Stability definitions
- Simulation in Matlab/Simulink

Material

- ▶ Glad& Ljung Ch. 11, 12.1, (Khalil Ch 2.3, part of 4.1, and 4.3) (Slotine and Li, pp 40-57)
- Lecture slides

Example - Linearization around equilibrium point

The linearization of

$$\ddot{x}(t) = \frac{g}{l}\sin x(t)$$

around the equilibrium $x_0 = n\pi$ is given by

$$\ddot{\tilde{x}}(t) = \frac{g}{l}\sin(n\pi + \tilde{x}(t)) \approx \frac{g}{l}(-1)^n \tilde{x}(t)$$

Hint: $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$

2 minute exercise: Linearize

$$\ddot{x} + \dot{x}^3 - \dot{x}^2 - x = u$$

around the solution

$$x_0(t) = e^t, \qquad u_0(t) = e^{3t} - e^{2t}$$

Hint: First check if (x_0, u_0) is a solution. Then plug-in $x(t) = e^t + \tilde{x}(t), u(t) = e^{3t} - e^{2t} + \tilde{u}(t)$, expand the expressions, and finally remove higher order terms (≥ 2) of \tilde{x} and \tilde{u} .

Linearization, cont'd

The linearization of the output equation

$$y(t) = h(x(t), u(t))$$

around the nominal output $y_0(t) = h(x_0(t), u_0(t))$ is given by

$$(y(t) - y_0(t)) = C(t)(x(t) - x_0(t)) + D(t)(u(t) - u_0(t))$$

where (if dim $y = \dim x = \dim u = 2$)

$$\begin{split} C(t) &= \left. \frac{\partial h}{\partial x} \right|_{(x_0,u_0)} = \left[\frac{\partial h_1}{\partial x_1} \cdot \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} \cdot \frac{\partial h_2}{\partial x_2} \right] \left|_{(x_0(t),u_0(t))} \right. \\ D(t) &= \left. \frac{\partial h}{\partial u} \right|_{(x_0,u_0)} = \left[\frac{\partial h_1}{\partial u_1} \cdot \frac{\partial h_1}{\partial u_2} \\ \frac{\partial h_2}{\partial u_1} \cdot \frac{\partial h_2}{\partial u_2} \right] \left|_{(x_0(t),u_0(t))} \right. \end{split}$$

To be able to

- linearize, both around equilibria and trajectories,
- explain definitions of local and global stability,
- check local stability and local controllability at equilibria
- simulate in Simulink,

Linearization Around a Trajectory

Idea:

Make Taylor-expansion around a known solution $\{x_0(t), u_0(t)\}$ Neglect small terms (i.e., keep the linear terms, as these will locally dominate over the higher order terms). Let

 $dx_0/dt = f(x_0(t), u_0(t))$ be a known solution How will a small deviation $\{\tilde{x}, \tilde{u}\}$ from this solution behave?

$$\frac{d(x_0+\tilde{x})}{dt} = f(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$$

$$(x_0(t), u_0(t))$$

$$(x_0(t), u_0(t))$$

$$(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$$

Linearization, explicit form

Consider $\dot{x}(t) = f(x(t), u(t))$ and assume $\dot{x}_0 = f(x_0(t), u_0(t))$

The linearization around $(x_0(t), u_0(t))$ is given by

$$\frac{d}{dt}(x(t) - x_0(t)) = A(t) \cdot (x(t) - x_0(t)) + B(t) \cdot (u(t) - u_0(t))$$

where (if dim $x = \dim u = 2$)

$$\begin{aligned} A(t) &= \left. \frac{\partial f}{\partial x} \right|_{(x_0,u_0)} = \left[\frac{\partial f_1}{\partial x_1} \frac{\partial f_1}{\partial x_2} \right]_{(x_0(t),u_0(t))} \\ B(t) &= \left. \frac{\partial f}{\partial u} \right|_{(x_0,u_0)} = \left[\frac{\partial f_1}{\partial u_1} \frac{\partial f_1}{\partial u_2} \right]_{(x_0(t),u_0(t))} \end{aligned}$$

Linearization Around a Trajectory, cont.

Let $(x_0(t), u_0(t))$ denote a solution to $\dot{x} = f(x, u)$ and consider another solution $(x(t), u(t)) = (x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$:

 $(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$

State-space form

Hence, for small (\tilde{x}, \tilde{u}) , approximately

$$\dot{\tilde{x}}(t) = A(x_0(t), u_0(t))\tilde{x}(t) + B(x_0(t), u_0(t))\tilde{u}(t)$$

where (if dim
$$x = 2$$
, dim $u = 1$)

$$\begin{split} A(x_0(t), u_0(t)) &= \frac{\partial f}{\partial x}(x_0(t), u_0(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \Big|_{(x_0(t), u_0(t))} \\ B(x_0(t), u_0(t)) &= \frac{\partial f}{\partial u}(x_0(t), u_0(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_2} \end{bmatrix} \Big|_{(x_0(t), u_0(t))} \end{split}$$

Note that *A* and *B* are **time dependent**! However, if we don't linearize around a trajectory but linearize around an equilibrium point $(x_0(t), u_0(t)) \equiv (x_0, u_0)$ then *A* and *B* are constant.

Example: Rocket

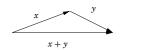
$$\dot{h}(t) = v(t)$$

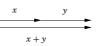
$$\dot{v}(t) = -g + \frac{v_e u(t)}{m(t)}$$

$$\dot{n}(t) = -u(t)$$
Let $u_0(t) \equiv u_0 > 0$; $x_0(t) = \begin{bmatrix} h_0(t) \\ v_0(t) \\ m_0(t) \end{bmatrix}$; $m_0(t) = m_0 - u_0 t$.
Linearization: $\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{-v_e u_0}{m_0(t)^2} \\ 0 & 0 & 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0 \\ \frac{v_e}{m_0(t)} \\ -1 \end{bmatrix} \tilde{u}(t)$

Definition: A *norm function* $||x|| : x \in \mathbb{R}^n \to \mathbb{R}$ satisfies the following three properties:

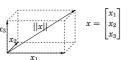
- ||x|| = 0 if and only if (iff) x = 0, ||x|| > 0 otherwise.
- ||ax|| = a||x||, for any positive *a* and any signal vector *x*.
- (The triangle inequality) $||x + y|| \le ||x|| + ||y||$





Definition: (Euclidean norm)

 $||x|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$



Asymptotic Stability

Definition The equilibrium x = 0 is locally asymptotically stable (LAS) if it

1) is stable

2) there exists r > 0 so that if ||x(0)|| < r then

$$x(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

(PhD-exercise: Show that 1) does not follow from 2))

Linearization, cont'd

The linearization of the output equation

$$y(t) = h(x(t), u(t))$$

around the nominal output $y_0(t) = h(x_0(t), u_0(t))$ is given by

$$(y(t) - y_0(t)) = C(t)(x(t) - x_0(t)) + D(t)(u(t) - u_0(t))$$

where (if dim $y = \dim x = 2$, dim u = 1)

$$C(t) = \frac{\partial h}{\partial x}\Big|_{(x_0,u_0)} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix}\Big|_{(x_0(t),u_0(t))}$$
$$D(t) = \frac{\partial h}{\partial u}\Big|_{(x_0,u_0)} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} \\ \frac{\partial h_2}{\partial u_1} \end{bmatrix}\Big|_{(x_0(t),u_0(t))}$$

Part II: Stability definitions

Local Stability

Consider $\dot{x} = f(x)$ where f(0) = 0

Definition The equilibrium x = 0 is **stable** if, for any R > 0, there exists r > 0, such that

$$||x(0)|| < r \implies ||x(t)|| < R$$
, for all $t \ge 0$

Otherwise the equilibrium point is unstable.



Global Asymptotic Stability

Definition The equilibrium is said to be **globally asymptotically stable (GAS)** if it is LAS and for all x(0) one has

 $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Part III: Check local stability and controllability

$\dot{x}_1 = -x_1^2 + x_1 + \sin(x_2)$

at
$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
 gives $A = \begin{pmatrix} -1 & 1 \\ -3 & -5 \end{pmatrix}$

Eigenvalues are given by the characteristic equation

$$0 = \det(\lambda I - A) = (\lambda + 1)(\lambda + 5) + 3$$

 $\dot{x}_2 = \cos(x_2) - x_1^3 - 5x_2$

Example

This gives $\lambda = \{-2, -4\}$, which are both in the left half-plane, hence the *nonlinear system* is LAS around x_0 .

5 minute exercise:

The linearization of

Is the ball and beam

$$\frac{7}{5}\ddot{x} = x\dot{\phi}^2 + g\sin\phi + \frac{2r}{5}\ddot{\phi}$$

nonlinearly locally controllable around $\dot{\phi} = \phi = x = \dot{x} = 0$ (with $\ddot{\phi}$ as input)?

Remark: This is a bit bit more detailed model of the ball and beam than we saw in Lecture 1.

However...

And now for the major limitation: The system works only in situations where the car can continuously back up into a space — not for those tight spots where you must inch your way into a space by going back and forth, wrestling with the wheel.

Unfortunately, such spots are quite common in Japan. And that's precisely when you wish you had a smart car that would graciously help you park. For me, the parking system also took some getting used to.

You can't turn the car too much before you start parking because the car will get confused and tell you to start over. You must decisively glide straight into pre-parking position before the car will let you begin jigging the arrows on the panel.

arrows on the panel. When I tried the system in our tiny parking lot at home, the system kept flashing warnings on the screen that the car was too close or too far from where I wanted to park.

Bosch 2008 (Automatic parking assistance)

- Multiple turns
- parking lot > car length + 80 cm

More parking in lecture 12

Lyapunov's Linearization Method

Theorem Assume

 $\dot{x} = f(x)$

has the linearization

$$\frac{d}{dt}(x(t) - x_0) = A(x(t) - x_0)$$

around the equilibrium point x_0 and put

$$\alpha(A) = \max \mathsf{Re}(\lambda(A))$$

- If $\alpha(A) < 0$, then $\dot{x} = f(x)$ is LAS at x_0 ,
- If $\alpha(A) > 0$, then $\dot{x} = f(x)$ is unstable at x_0 ,
- If $\alpha(A) = 0$, then no conclusion can be drawn.

(Proof in Lecture 4)

Local Controllability

Theorem Assume

$$\dot{x} = f(x, u)$$

has the linearization

$$\frac{d}{dt}(x(t) - x_0) = A(x(t) - x_0) + B(u(t) - u_0)$$

around the equilibrium (x_0, u_0) then

• (A, B) controllable $\Rightarrow f(x, u)$ nonlinear locally controllable

Here nonlinear locally controllable is defined as:

For every T > 0 and $\varepsilon > 0$ the set of states x(T) that can be reached from $x(0) = x_0$, by using controls satisfying $||u(t) - u_0|| < \varepsilon$, contains a small ball around x_0 .

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| POLITICS | By YURI KAGEYAMA THE ASSOCIATED PRESS Last Updated: January 16, 2004, 10:00:12 AM PST TOKYO Your hands don't even need to be touching the steering wheel for it to star spinning back and forth aggressively, all by itself- slowly guiding the car in to the parking spit. Parallel parking is designed to be a breeze with the Intelligent Parking Assist system, part of a new 52,200 option package for Toyota Motor Cong's Finas gas-leckric hybridin i Japan. This is a bold and somewhat tumerving concept, a car that parks itself As a driver, you've gat to wonder as the Prius cases back toward the curb. What it his machine thinking? It is als official not to be parallel parking. Toyota has reased an Intelligent Parking Assist that allows a vehicle to park hybrid is in parkine thinking? It is als official not to be gartial parking is a parallel. | | | | | |
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| LIFE Monday Life Technology Healthy Living Taste Buzzz | | | | | | |
| Leisure & Style Wheels Faith & Values Friends & Family Travel Your Home Horoscope | | | | | | |
| THE ARTS | But we'll get to the drawbacks later. \$2,200 option, however, is only available so far on Toyota's Prius | | | | | |
| BUSINESS | Direct the laste behind this incorrection | gas-electric hybrid in Japan, and has | | | | |
| COLUMNISTS | First, the logic behind this innovation. other limitations that make it far find the perfect hands-free parking | | | | | |
| GALLERIES | If you know Japan, it should come as no surprise that about 80 percent THE ASSOCIATED PRESS | | | | | |
| VALLEY MALL | of its Prius buyers have opted for Intelligent Parking Assist. This is a | | | | | |

Example

An inverted pendulum with vertically moving pivot point



$$\ddot{\phi}(t) = \frac{1}{t} \left(g + u(t) \right) \sin(\phi(t)),$$

where u(t) is acceleration, can be written as

$$\dot{x}_1 = x_2$$

 $\dot{x}_2 = \frac{1}{l}(g+u)\sin(x_1)$

Example, cont.

The linearization around $x_1 = x_2 = 0, u = 0$ is given by

$$\dot{x}_1 = x_2$$

 $\dot{x}_2 = rac{g}{l} x_1$

It is not controllable, hence no conclusion can be drawn about nonlinear controllability

However, simulations show that the system is stabilized by

$$u(t) = \varepsilon \omega^2 \sin(\omega t)$$

if ω is large enough !

Demonstration We will come back to this example later.

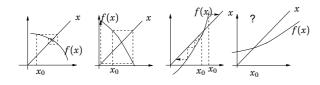
Example (cont'd): Numerical iteration

$$x_{k+1} = f(x_k)$$

to find fixed point

 $x^* = f(x^*)$

When does the iteration converge?



Simulink

> matlab >> simulink



Choose Simulation Parameters

| Iver Workspace I/O Diagno | |
|---------------------------|--------------------------|
| Start time: 0.0 | Stop time: 5.0 |
| Solver options | |
| Type: Variable-step 🖃 | ode45 (Dormand-Prince) 💴 |
| Max step size: auto | Relative tolerance: 1e-3 |
| Initial step size: auto | Absolute tolerance: 1e-6 |
| | |
| Output options | |
| Refine output | Refine factor: 1 |

Don't forget "Apply"

Bonus — Discrete Time

Many results are parallel (observability, controllability,...)

Example: The difference equation

$$x_{k+1} = f(x_k)$$

is asymptotically stable at x^* if the linearization

$$\left. \frac{\partial f}{\partial x} \right|_{x^*}$$
 has all eigenvalues in $|\lambda| < 1$

(that is, within the unit circle).

Part IV: Simulation

Often the only method

- $\dot{x} = f(x)$
 - ACSL
 - Simnon
 - Simulink
- $F(\dot{x},x)=0$
 - Omsim http://www.control.lth.se/~cace/omsim.html
- Dymola http://www.dynasim.se/
- Modelica

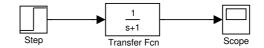
http://www.dynasim.se/Modelica/index.html Special purpose

Spice (electronics)

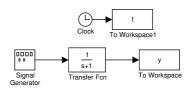
- EMTP (electromagnetic transients)
- Adams (mechanical systems)

Simulink, An Example

File -> New -> Model
Double click on Continuous
Transfer Fcn
Step (in Sources)
Scope (in Sinks)
Connect (mouse-left)
Simulation->Parameters



Save Results to Workspace



Check "Save format" of output blocks ("Array" instead of "Structure") >> plot(t,y)

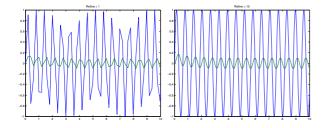
(or use "Structure" which also contains the time information.)

How To Get Better Accuracy

Use Scripts to Document Simulations

Modify Refine, Absolute and Relative Tolerances, Integration method

Refine adds interpolation points:



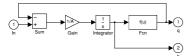
Submodels, Example: Water tanks

Equation for one water tank:

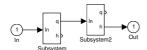
$$\dot{h} = (u-q)/A$$

 $q = a\sqrt{2g}\sqrt{h}$

Corresponding Simulink model:



Make a subsystem and connect two water tanks in series.



Linearization in Simulink, cont.

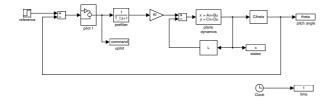
Use the command linmod to find a linear approximation of the system around an operating point:

>> [aa,bb,cc,dd]=linmod('flow',x0,u0);
>> sys=ss(aa,bb,cc,dd);

>> bode(sys)

Computer exercise

Simulation of JAS 39 Gripen



- Simulation
- Analysis of PIO using describing functions
- Improve design

If the block-diagram is saved to stepmodel.mdl, the following Script-file simstepmodel.m simulates the system:

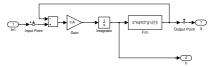
open_system('stepmodel')
set_param('stepmodel', 'RelTol', '1e-3')
set_param('stepmodel', 'AbsTol', '1e-6')
set_param('stepmodel', 'Refine', '1')
tic
sim('stepmodel', 6)
toc
subplot(2,1,1),plot(t,y),title('y')
subplot(2,1,2),plot(t,u),title('u')

Linearization in Simulink

Use the command trim to find *e.g.*, stationary points to a system >> A=2.7e-3;a=7e-6,g=9.8;

Linearization in Simulink; Alternative

By right-clicking on a signal connector in a Simulink model you can add "Linearization points" (inputs and/or outputs).



Start a "Control and Estimation Tool Manager" to get a linearized model by

Tools -> Control Design ->Linear analysis ...

where you can set the operating points, export linearized model to Workspace (Model-> Export to Workspace) and much more.

Today

- Linearization, both around equilibria and trajectories,
- Definitions of local and global stability,
- How to check local stability and local controllability at equilibria
- Simulation tool: Simulink,

Next Lecture

- Phase plane analysis
- Classification of equilibria