

Input-Output Models

Real-Time Systems, Lecture 7

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Lecture 7: Input-Output Models

[IFAC PB Ch 3 p 22-34]

- Shift operators; the pulse transfer operator
- Z-transform; the pulse transfer function
- Transformations between system representations
- System response, frequency response
- ZOH sampling of a transfer function

1

Linear System Models

	State-space model	Input-output models	
		Differential/difference equation	Transfer operator/fcn
CT	$\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t)$	$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y$ $= b_1 \frac{d^{n-1} u}{dt^{n-1}} + \dots + b_n u$	$G(p) / G(s)$
DT	$x(k+1) = \Phi x(k) + \Gamma u(k)$ $y(k) = Cx(k)$	$y(k) + a_1 y(k-1) + \dots +$ $a_n y(k-n) = b_1 u(k-1) +$ $\dots + b_n u(k-n)$	$H(q) / H(z)$

More I-O models: (im)pulse response, step response, frequency response

2

Shift Operators

Operators on time series

The sampling period is chosen as the time unit ($f(k) \Leftrightarrow f(kh)$)

Time series are doubly infinite sequences:

- $f(k) : k = \dots -1, 0, 1, \dots$

Forward shift operator:

- denoted q
- $qf(k) = f(k+1)$
- $q^n f(k) = f(k+n)$

Backward shift operator:

- denoted q^{-1}
- $q^{-1}f(k) = f(k-1)$
- $q^{-n}f(k) = f(k-n)$

3

Pulse Transfer Operator

Rewrite the state-space model using the forward shift operator:

$$\begin{aligned} x(k+1) &= qx(k) = \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Eliminate $x(k)$:

$$\begin{aligned} x(k) &= (qI - \Phi)^{-1} \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) = C(qI - \Phi)^{-1} \Gamma u(k) + Du(k) \\ &= [C(qI - \Phi)^{-1} \Gamma + D] u(k) = H(q)u(k) \end{aligned}$$

$H(q)$ is the *pulse transfer operator* of the system

Describes how the input and output are related.

4

Poles and Zeros (SISO case)

The pulse transfer function is a rational function

$$H(q) = \frac{B(q)}{A(q)}$$

$\deg A = n$ = the number of states

$\deg B = n_b \leq n$

$A(q)$ is the characteristic polynomial of Φ , i.e.

$$A(q) = \det(qI - \Phi)$$

The *poles* of the system are given by $A(q) = 0$

The *zeros* of the system are given by $B(q) = 0$

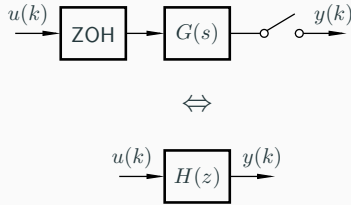
5

<div data-bbox="38 262 429 293" data-label="Section-Header"> <h3>Interpretation of Poles and Zeros</h3> </div> <div data-bbox="78 342 129 367" data-label="Text"> <p>Poles:</p> </div> <div data-bbox="98 387 644 416" data-label="List-Group"> <ul style="list-style-type: none"> • A pole in a is associated with the time function $f(k) = a^k$ </div> <div data-bbox="78 461 129 486" data-label="Text"> <p>Zeros:</p> </div> <div data-bbox="98 506 716 600" data-label="List-Group"> <ul style="list-style-type: none"> • A zero in a implies that the transmission of the input $u(k) = a^k$ is blocked by the system • Related to how inputs and outputs are coupled to the states </div> <div data-bbox="126 629 684 754" data-label="Figure"> </div> <div data-bbox="772 792 783 813" data-label="Text"> <p>6</p> </div>	<div data-bbox="818 262 1050 293" data-label="Section-Header"> <h3>Disk Drive Example</h3> </div> <div data-bbox="858 356 1342 380" data-label="Text"> <p>Recall the double integrator from the previous lecture:</p> </div> <div data-bbox="1074 400 1307 506" data-label="Equation-Block"> $\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ $y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$ </div> <div data-bbox="858 560 1032 584" data-label="Text"> <p>Sample with $h = 1$:</p> </div> <div data-bbox="1067 604 1307 745" data-label="Equation-Block"> $\Phi = e^{Ah} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\Gamma = \int_0^h e^{As} B ds = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ </div> <div data-bbox="1554 792 1565 813" data-label="Text"> <p>7</p> </div>
<div data-bbox="38 844 269 875" data-label="Section-Header"> <h3>Disk Drive Example</h3> </div> <div data-bbox="78 904 287 929" data-label="Text"> <p>Pulse transfer operator:</p> </div> <div data-bbox="158 940 654 1050" data-label="Equation-Block"> $H(q) = C(qI - \Phi)^{-1} \Gamma + D$ $= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q-1 & -1 \\ 0 & q-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \frac{0.5(q+1)}{(q-1)^2}$ </div> <div data-bbox="78 1068 360 1093" data-label="Text"> <p>Two poles in 1, one zero in -1.</p> </div> <div data-bbox="228 1093 588 1379" data-label="Figure"> </div> <div data-bbox="772 1375 783 1395" data-label="Text"> <p>8</p> </div>	<div data-bbox="818 844 1449 875" data-label="Section-Header"> <h3>From Pulse Transfer Operator to Difference Equation</h3> </div> <div data-bbox="1086 1008 1284 1070" data-label="Equation-Block"> $y(k) = H(q)u(k)$ $A(q)y(k) = B(q)u(k)$ </div> <div data-bbox="927 1113 1449 1142" data-label="Equation-Block"> $(q^n + a_1 q^{n-1} + \dots + a_n)y(k) = (b_0 q^{n_b} + \dots + b_{n_b})u(k)$ </div> <div data-bbox="858 1167 973 1191" data-label="Text"> <p>which means</p> </div> <div data-bbox="922 1216 1453 1279" data-label="Equation-Block"> $y(k+n) + a_1 y(k+n-1) + \dots + a_n y(k) = b_0 u(k+n_b) + \dots + b_{n_b} u(k)$ </div> <div data-bbox="1554 1375 1565 1395" data-label="Text"> <p>9</p> </div>
<div data-bbox="38 1426 517 1458" data-label="Section-Header"> <h3>Difference Equation with Backward Shift</h3> </div> <div data-bbox="140 1612 670 1677" data-label="Equation-Block"> $y(k+n) + a_1 y(k+n-1) + \dots + a_n y(k) = b_0 u(k+n_b) + \dots + b_{n_b} u(k)$ </div> <div data-bbox="78 1702 233 1727" data-label="Text"> <p>can be written as</p> </div> <div data-bbox="140 1749 670 1814" data-label="Equation-Block"> $y(k) + a_1 y(k-1) + \dots + a_n y(k-n) = b_0 u(k-d) + \dots + b_{n_b} u(k-d-n_b)$ </div> <div data-bbox="78 1836 531 1861" data-label="Text"> <p>where $d = n - n_b$ is the <i>pole excess</i> of the system.</p> </div> <div data-bbox="772 1955 783 1975" data-label="Text"> <p>10</p> </div>	<div data-bbox="818 1426 1299 1458" data-label="Section-Header"> <h3>Difference Equation with Backward Shift</h3> </div> <div data-bbox="858 1563 1091 1588" data-label="Text"> <p>The <i>reciprocal polynomial</i></p> </div> <div data-bbox="994 1612 1380 1641" data-label="Equation-Block"> $A^*(q) = 1 + a_1 q + \dots + a_n q^n = q^n A(q^{-1})$ </div> <div data-bbox="858 1664 1430 1720" data-label="Text"> <p>is obtained from the polynomial A by reversing the order of the coefficients.</p> </div> <div data-bbox="858 1785 1235 1809" data-label="Text"> <p>Now the system can instead be written as</p> </div> <div data-bbox="1037 1832 1337 1861" data-label="Equation-Block"> $A^*(q^{-1})y(k) = B^*(q^{-1})u(k-d)$ </div> <div data-bbox="1554 1955 1565 1975" data-label="Text"> <p>11</p> </div>

Difference Equation Example	Z-transform																								
<p>Using forward shift</p> $y(k+2)+2y(k+1)+3y(k)=2u(k+1)+u(k)$ <p>can be written</p> $(q^2+2q+3)y(k)=(2q+1)u(k)$ <p>Hence,</p> $A(q)=q^2+2q+3$ $B(q)=2q+1$ <p>Using backward shift, the same equation can be written ($d=1$)</p> $(1+2q^{-1}+3q^{-2})y(k)=(2+q^{-1})u(k-1)$ <p>Hence,</p> $A^*(q^{-1})=1+2q^{-1}+3q^{-2}$ $B^*(q^{-1})=2+q^{-1}$	<p>The discrete-time counterpart to the Laplace transform</p> <p>Defined on semi-infinite time series $f(k):k=0,1,\dots$</p> $\mathcal{Z}\{f(k)\}=F(z)=\sum_{k=0}^{\infty}f(k)z^{-k}$ <p>z is a complex variable</p>																								
Example – Discrete-Time Step Signal	Z-transform Table																								
<p>Let $y(k)=1$ for $k\geq 0$. Then</p> $Y(z)=1+z^{-1}+z^{-2}+\dots=\frac{z}{z-1},\quad z >1$ <p>Application of the following result for power series</p> $\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}\text{ for } x <1$	<p>Table 2 (p 26) in IFAC PB (ignore the middle column!)</p> <table><tr><th>f</th><th>$\mathcal{L}f$</th><th>$\mathcal{Z}f$</th></tr><tr><td>$\delta(k)$ (pulse)</td><td>–</td><td>1</td></tr><tr><td>1 $k\geq 0$ (step)</td><td>$\frac{1}{s}$</td><td>$\frac{z}{z-1}$</td></tr><tr><td>kh</td><td>$\frac{1}{s^2}$</td><td>$\frac{hz}{(z-1)^2}$</td></tr><tr><td>$\frac{1}{2}(kh)^2$</td><td>$\frac{1}{s^3}$</td><td>$\frac{h^2z(z+1)}{2(z-1)^3}$</td></tr><tr><td>$e^{-kh/T}$</td><td>$\frac{T}{1+sT}$</td><td>$\frac{z}{z-e^{-h/T}}$</td></tr><tr><td>$1-e^{-kh/T}$</td><td>$\frac{1}{s(1+sT)}$</td><td>$\frac{z(1-e^{-h/T})}{(z-1)(z-e^{-h/T})}$</td></tr><tr><td>$\sin\omega kh$</td><td>$\frac{\omega}{s^2+\omega^2}$</td><td>$\frac{z\sin\omega h}{z^2-2z\cos\omega h+1}$</td></tr></table>	f	$\mathcal{L}f$	$\mathcal{Z}f$	$\delta(k)$ (pulse)	–	1	1 $k\geq 0$ (step)	$\frac{1}{s}$	$\frac{z}{z-1}$	kh	$\frac{1}{s^2}$	$\frac{hz}{(z-1)^2}$	$\frac{1}{2}(kh)^2$	$\frac{1}{s^3}$	$\frac{h^2z(z+1)}{2(z-1)^3}$	$e^{-kh/T}$	$\frac{T}{1+sT}$	$\frac{z}{z-e^{-h/T}}$	$1-e^{-kh/T}$	$\frac{1}{s(1+sT)}$	$\frac{z(1-e^{-h/T})}{(z-1)(z-e^{-h/T})}$	$\sin\omega kh$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin\omega h}{z^2-2z\cos\omega h+1}$
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Some Properties of the Z-transform	From State Space to Pulse Transfer Function																								
$\mathcal{Z}(\alpha f+\beta g)=\alpha F(z)+\beta G(z)$ $\mathcal{Z}(q^{-n}f)=z^{-n}F(z)$ $\mathcal{Z}(qf)=z(F(z)-f(0))$ $\mathcal{Z}(f*g)=\mathcal{Z}\left\{\sum_{j=0}^kf(j)g(k-j)\right\}=F(z)G(z)$	$\begin{cases} x(k+1)=\Phi x(k)+\Gamma u(k) \\ y(k)=Cx(k)+Du(k) \end{cases}$ $\begin{cases} z(X(z)-x(0))=\Phi X(z)+\Gamma U(z) \\ Y(z)=CX(z)+DU(z) \end{cases}$ $Y(z)=C(zI-\Phi)^{-1}zx(0)+[C(zI-\Phi)^{-1}\Gamma+D]U(z)$ <p>The rational function $H(z)=C(zI-\Phi)^{-1}\Gamma+D$ is called the <i>pulse transfer function</i> from u to y.</p> <p>It is the Z-transform of the pulse response $h(k)$</p>																								

<div data-bbox="36 257 188 293" data-label="Section-Header"> <h3>$H(q)$ vs $H(z)$</h3> </div> <div data-bbox="76 383 711 436" data-label="Text"> <p>The pulse transfer operator $H(q)$ and the pulse transfer function $H(z)$ are the same rational functions</p> </div> <div data-bbox="76 477 402 504" data-label="Text"> <p>They have the same poles and zeros</p> </div> <div data-bbox="98 521 600 586" data-label="List-Group"> <ul style="list-style-type: none"> $H(q)$ is used in the time domain (q = shift operator) $H(z)$ is used in the Z-domain (z = complex variable) </div> <div data-bbox="76 629 705 716" data-label="Text"> <p>If the order of $H(q)$ or $H(z)$ is less than the order of the original state-space system, i.e., a pole-zero cancellation has taken place, then the system is either not reachable or not observable</p> </div> <div data-bbox="764 792 783 813" data-label="Page-Footer"> <p>18</p> </div>	<div data-bbox="818 257 1436 293" data-label="Section-Header"> <h3>Calculating System Response Using the Z-transform</h3> </div> <div data-bbox="873 421 1445 546" data-label="List-Group"> <ol style="list-style-type: none"> Find the pulse transfer function $H(z) = C(zI - \Phi)^{-1}\Gamma + D$ Compute the Z-transform of the input: $U(z) = \mathcal{Z}\{u(k)\}$ Compute the Z-transform of the output: </div> <div data-bbox="1026 566 1391 595" data-label="Equation-Block"> $Y(z) = C(zI - \Phi)^{-1}z x(0) + H(z)U(z)$ </div> <div data-bbox="873 618 1412 678" data-label="List-Group"> <ol style="list-style-type: none"> Apply the inverse Z-transform (table) to find the output: $y(k) = \mathcal{Z}^{-1}\{Y(z)\}$ </div> <div data-bbox="1544 792 1564 813" data-label="Page-Footer"> <p>19</p> </div>
<div data-bbox="36 842 510 875" data-label="Section-Header"> <h3>Frequency Response – Continuous Time</h3> </div> <div data-bbox="220 898 659 1099" data-label="Figure"> </div> <div data-bbox="76 1120 649 1173" data-label="Text"> <p>Given a stable system $G(s)$, the input $u(t) = \sin \omega t$ will, after a transient, give the output</p> </div> <div data-bbox="240 1187 568 1229" data-label="Equation-Block"> $y(t) = G(j\omega) \sin(\omega t + \arg G(j\omega))$ </div> <div data-bbox="98 1261 726 1350" data-label="List-Group"> <ul style="list-style-type: none"> The amplitude and phase shift for different frequencies are given by the value of $G(s)$ along the imaginary axes, i.e. $G(j\omega)$ Plotted in Bode and Nyquist diagrams </div> <div data-bbox="764 1373 783 1393" data-label="Page-Footer"> <p>20</p> </div>	<div data-bbox="898 931 1473 1055" data-label="Figure"> </div> <div data-bbox="858 1099 1455 1155" data-label="Text"> <p>Given a stable system $H(z)$, the input $u(k) = \sin(\omega k)$ will, after a transient, give the output</p> </div> <div data-bbox="1010 1171 1362 1214" data-label="Equation-Block"> $y(k) = H(e^{j\omega}) \sin(\omega k + \arg H(e^{j\omega}))$ </div> <div data-bbox="880 1252 1517 1350" data-label="List-Group"> <ul style="list-style-type: none"> $G(s)$ and the imaginary axis are replaced by $H(z)$ and the unit circle. Only describes what happens at the sampling instants The inter-sample behavior is not studied in this course </div> <div data-bbox="1544 1373 1564 1393" data-label="Page-Footer"> <p>21</p> </div>
<div data-bbox="36 1424 207 1458" data-label="Section-Header"> <h3>Bode Diagram</h3> </div> <div data-bbox="76 1485 705 1541" data-label="Text"> <p>Bode diagram for $G(s) = 1/(s^2 + 1.4s + 1)$ (solid) and ZOH-sampled counterpart $H(z)$ (dashed, plotted for $\omega h \in [0, \pi]$)</p> </div> <div data-bbox="188 1543 625 1928" data-label="Figure"> </div> <div data-bbox="76 1948 577 1975" data-label="Text"> <p>The hold circuit can be approximated by a delay of $h/2$</p> </div> <div data-bbox="764 1955 783 1975" data-label="Page-Footer"> <p>22</p> </div>	<div data-bbox="818 1424 1018 1458" data-label="Section-Header"> <h3>Nyquist Diagram</h3> </div> <div data-bbox="858 1494 1511 1552" data-label="Text"> <p>Nyquist diagram for $G(s) = 1/(s^2 + 1.4s + 1)$ (solid) and ZOH-sampled counterpart $H(z)$ (dashed, plotted for $\omega h \in [0, \pi]$)</p> </div> <div data-bbox="978 1556 1399 1937" data-label="Figure"> </div> <div data-bbox="1544 1955 1564 1975" data-label="Page-Footer"> <p>23</p> </div>

ZOH Sampling of a Transfer Function



How to calculate $H(z)$ given $G(s)$?

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Calculation of $H(z)$ Given $G(s)$

Three approaches:

1. Make a state-space realization of $G(s)$. Sample using ZOH to obtain Φ and Γ . Then $H(z) = C(zI - \Phi)^{-1}\Gamma + D$.
 - Works also for systems with time delays, $G(s)e^{-s\tau}$
2. Use the formula

$$H(z) = \frac{z-1}{z} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sh}}{z - e^{sh}} \frac{G(s)}{s} ds$$

$$= \sum_{s=s_i} \frac{1}{z - e^{sh}} \text{Res} \left\{ \frac{e^{sh} - 1}{s} G(s) \right\}$$

- s_i are the poles of $G(s)$ and Res denotes the residue.
- outside the scope of the course

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Calculation of $H(z)$ Given $G(s)$

3. Use Table 3 (p 27) in IFAC PB

$G(s)$	$H(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n}{z^n + a_1 z^{n-1} + \dots + a_n}$
$\frac{1}{s}$	$\frac{h}{z-1}$
$\frac{1}{s^2}$	$\frac{h^2(z+1)}{2(z-1)^2}$
e^{-sh}	z^{-1}
$\frac{a}{s+a}$	$\frac{1 - \exp(-ah)}{z - \exp(-ah)}$
$\frac{a}{s(s+a)}$	$b_1 = \frac{1}{a}(ah - 1 + e^{-ah})$ $b_2 = \frac{1}{a}(1 - e^{-ah} - ah e^{-ah})$ $a_1 = -(1 + e^{-ah})$ $a_2 = e^{-ah}$

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Calculation of $H(z)$ Given $G(s)$

Example: For $G(s) = e^{-\tau s}/s^2$, the previous lecture gave

$$x(kh + h) = \Phi x(kh) + \Gamma_1 u(kh - h) + \Gamma_0 u(kh)$$

$$\Phi = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \quad \Gamma_1 = \begin{pmatrix} \tau \left(h - \frac{\tau}{2} \right) \\ \tau \end{pmatrix} \quad \Gamma_0 = \begin{pmatrix} \frac{(h-\tau)^2}{2} \\ h - \tau \end{pmatrix}$$

With $h = 1$ and $\tau = 0.5$, this gives

$$H(z) = C(zI - \Phi)^{-1}(\Gamma_0 + \Gamma_1 z^{-1}) = \frac{0.125(z^2 + 6z + 1)}{z(z^2 - 2z + 1)}$$

Order: 3

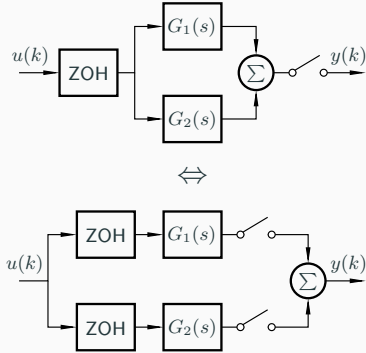
Poles: 0, 1, and 1

Zeros: $-3 \pm \sqrt{8}$

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Calculation of $H(z)$ Given $G(s)$

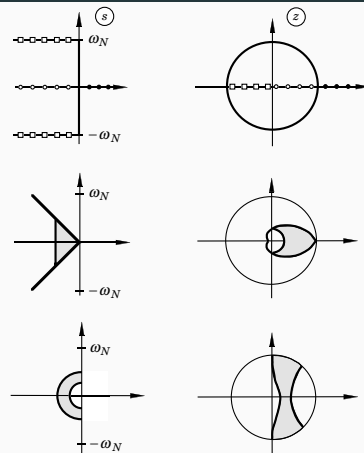
ZOH sampling is a linear operation, so a transfer function $G(s)$ may be split into smaller parts $G_1(s) + G_2(s) + \dots$ that are sampled separately



This does not hold for series decomposition, i.e., $\text{ZOH}(G_1(s)G_2(s)) \neq \text{ZOH}(G_1(s))\text{ZOH}(G_2(s))$

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Transformation of Poles via ZOH Sampling: $z_i = e^{s_i h}$

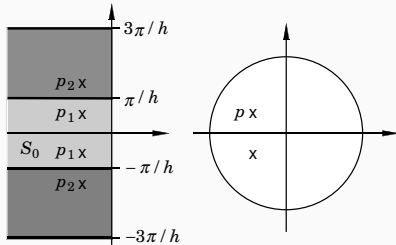


Note: The stability properties are preserved by ZOH sampling!

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New Evidence of the Alias Problem

Several points in the s -plane are mapped into the same point in the z -plane. The map is not bijective



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Transformation of Zeros via Sampling

- More complicated than for poles
- Extra zeros may appear in the sampled system
- There can be zeros outside the unit circle (non-minimum phase) even if the continuous system has all the zeros in the left half plane
- For short sampling periods

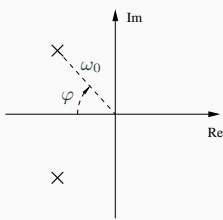
$$z_i \approx e^{s_i h}$$

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ZOH Sampling of a Second Order System

Second order continuous-time system with complex poles:

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad \zeta < 1$$



- Larger $\omega_0 \Rightarrow$ faster system response
- Smaller $\varphi \Rightarrow$ larger damping. Relative damping $\zeta = \cos \varphi$.
 - Common control design choice: $\zeta = \cos 45^\circ \approx 0.7$

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Sampled Second Order System

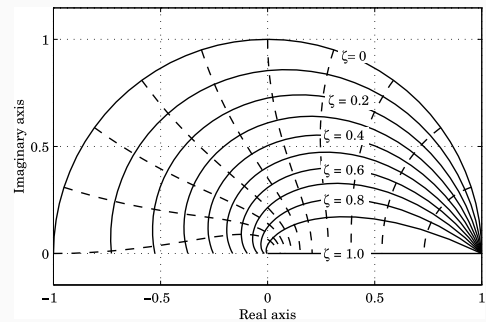
The poles of the sampled system are given by

$$z^2 + a_1 z + a_2 = 0$$

where

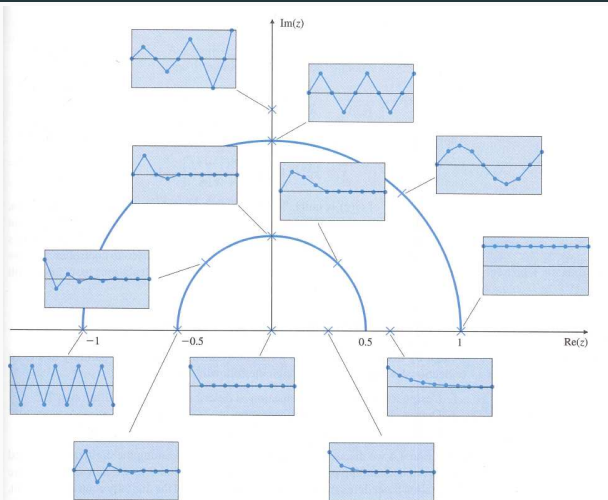
$$a_1 = -2e^{-\zeta\omega_0 h} \cos(\sqrt{1-\zeta^2}\omega_0 h)$$

$$a_2 = e^{-2\zeta\omega_0 h}$$



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Sampled Second Order System



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Examples in Matlab

```
>> % From state space system to pulse transfer function
>> A = [0 1; 0 0];
>> B = [0; 1];
>> C = [1 0];
>> D = 0;
>> contsys = ss(A,B,C,D);
>> h = 1;
>> discsys = c2d(contsys,h);
>> tf(discsys) % pulse transfer function
>> zpk(discsys) % factored pulse transfer function

>> % Bode and Nyquist diagrams
>> s = tf('s'); G = 1/(s^2+1.4*s+1);
>> H = c2d(G,1);
>> bode(G,H)
>> nyquist(G,H)

>> % Sampling of a second-order transfer function
>> G = 1/(s^2+s+1);
>> h = 0.1;
>> H = c2d(G,h)
>> zpk(H)
```

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