Updated solutions to Problems 2.4, 4.1, and 6.1

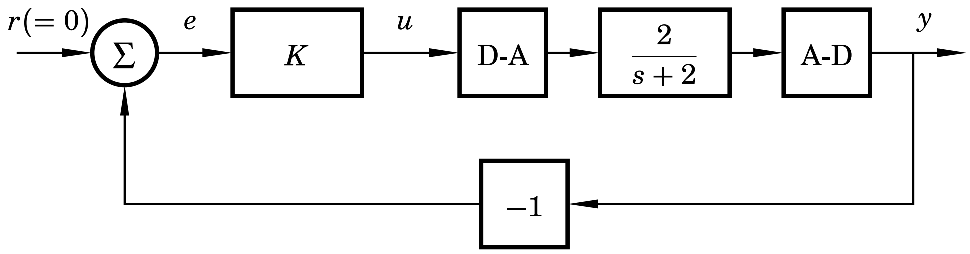

Figure 2.1 Computer-controlled system.

2.4 Consider the computer-controlled system in Fig. 2.1. The P-controller should execute with the sampling interval $h = 0.5$ s, and the controller gain is given by $K = 2$.

a. Consider the following implementation of the controller:

```
LOOP
  y = readInput();
  u = -K*y;
  writeOutput(u);
  waitForNextPeriod();
END;
```

Assume that the execution time of the controller is negligible. Show that the closed-loop system is stable.

b. Now consider the following implementation of the controller:

```
LOOP
  writeOutput(u);
  y = readInput();
  u = -K*y;
  waitForNextPeriod();
END;
```

Will the closed-loop system still be stable?

c. Consider again the implementation from subproblem a, i.e.:

```
LOOP
  y = readInput();
  u = -K*y;
  writeOutput(u);
  waitForNextPeriod();
END;
```

Assume now that the execution time of the controller is not negligible, i.e. that there is a delay, L, between `readInput()` and `writeOutput(u)`. Assuming that L is constant and $L < h$, what is the largest value of L for the system to be stable?

Hint: The stability conditions for a second-order discrete-time system with the characteristic polynomial $A(z) = z^2 + a_1 z + a_2$ are given by

$$\begin{aligned} a_2 &< 1 \\ a_2 &> -1 + a_1 \\ a_2 &> -1 - a_1 \end{aligned}$$

Solution

a. Sampling the process using the table "Zero-order hold sampling of a continuous-time system with transfer function $G(s)$" gives

$$H(z) = \frac{0.6321}{z - 0.3679}$$

The closed-loop system is given by

$$H_{cl}(z) = \frac{KH(z)}{1 + KH(z)} = \frac{1.264}{z + 0.8964}$$

The pole is located in -0.8964, inside the unit circle, so the closed-loop system is stable.

b. The sampled process, including a one sample delay, is now given by

$$H(z) = \frac{0.6321}{z(z - 0.3679)}$$

The closed-loop system is given by

$$H_{cl}(z) = \frac{1.264}{z^2 - 0.3679z + 1.264}$$

The poles are located in $0.1836 \pm 1.1092i$, i.e., outside the unit circle, so the closed-loop system in unstable.

c. We start by writing the continuous-time system on state-space form, i.e.,

$$\begin{aligned} dx(t)/dt &= -2x(t) + 2u(t) \\ y(t) &= x(t) \end{aligned}$$

The computational delay is equivalent to a constant input delay, i.e., the continuous-time system will be

$$\begin{aligned} dx(t)/dt &= -2x(t) + 2u(t - L) \\ y(t) &= x(t) \end{aligned}$$

The ZOH-sampled equivalent of this, assuming that $L \leq h$ is

$$\begin{aligned} x(kh + h) &= \Phi x(kh) + \Gamma_0 u(kh) + \Gamma_1 u(kh - h) \\ y(kh) &= x(kh) \end{aligned}$$

where

$$
\begin{aligned}
\Phi &= e^{-2h} = e^{-1} \\
\Gamma_0 &= 2\int_0^{h-L} e^{-2s} ds = 1 - e^{2L-1} \\
\Gamma_1 &= 2e^{-2(h-L)} \int_0^{L} e^{-2s} ds = e^{2L-1} - e^{-1}
\end{aligned}
$$

Applying the control law $u(k) = -2y(k) = -2x(k)$ gives the closed loop system

$$
x(k+1) = e^{-1}x(k) - 2(1-e^{2L-1})x(k) - 2(e^{2L-1} - e^{-1})x(k-1)
$$

The characteristic equation is hence

$$
z^2 + (2(1-e^{2L-1}) - e^{-1})z + 2(e^{2L-1} - e^{-1})
$$

Introducing $\omega = e^{2L-1}$, the conditions for stability can be written

$$
\begin{aligned}
&2(\omega - e^{-1}) < 1 \\
&2(\omega - e^{-1}) > -1 + (2(1-\omega) - e^{-1}) \\
&2(\omega - e^{-1}) > -1 - (2(1-\omega) - e^{-1})
\end{aligned}
$$

From this follows that

$$
\begin{aligned}
\omega &< \frac{1}{2} + e^{-1} \\
\omega &> \frac{1+e^{-1}}{4}
\end{aligned}
$$

The first inequality leads to

$$
2L - 1 < log(1/2 + e^{-1}) = -0.1417
$$

from which follows that

$$
L < 0.4291
$$

From the second inequality we have that

$$
2L - 1 > log(1/4 + e^{-1}/4) = -1.0730
$$

from which follows that

$$
L > -0.0365
$$

Hence, the largest delay L for which the system is stable is $L = 0.4291$.

4.1 Assuming the sampling interval h, use the various methods below to determine discrete-time approximations of the stable transfer function

$$
G(s) = \frac{a}{s+a}, \qquad a > 0
$$

For what values of h is the discrete-time system stable? For what values of h is the pole on the positive real axis? (What is the qualitative behavior of the discrete-time system if the pole is on the negative real axis?)

a. Forward difference (Euler's method)

b. Backward difference

c. Tustin's method

d. Tustin's method with prewarping and warping frequency $\omega_1 = a$ rad/s

Solution

a. Euler's method implies a translation

$$s' = \frac{z-1}{h}$$

We get

$$H(z) = G(s') = \frac{a}{\frac{z-1}{h} + a} = \frac{ah}{z + (ah-1)}$$

The discrete-time pole is located in $z = 1 - ah$. The system is stable if the pole is inside the unit circle. This is the case for $h < \frac{2}{a}$.

For $h < \frac{1}{a}$ the pole ends up on the positive real axis, and the discrete-time system will have a somewhat similar behavior to the continuous-time system. For $h > \frac{1}{a}$, the pole will be on the negative real axis, and the system will oscillate – completely unlike the continuous system.

b. The backward difference implies a translation

$$s' = \frac{z-1}{zh}$$

We get

$$H(z) = G(s') = \frac{a}{\frac{z-1}{zh} + a} = \frac{zah}{z(1+ah) - 1}$$

The discrete-time pole is located in $z = \frac{1}{1+ah}$. The system is stable and the pole is on the positive real axis for all values of h.

c. We use $s' = \frac{2}{h}\frac{z-1}{z+1}$ and obtain

$$H(z) = G(s') = \frac{a}{\frac{2}{h}\frac{z-1}{z+1} + a} = \frac{ah(z+1)}{2(z-1) + ah(z+1)} = \frac{ah(z+1)}{(2+ah)z + (ah-2)}$$

The discrete-time pole is located in $z = \frac{2-ah}{2+ah}$. The system is stable for all values of h, since

$$\left|\tfrac{2-ah}{2+ah}\right| < 1 \quad \Leftrightarrow \quad -2 < 2 < 2 + 2ah$$

which holds for all positive values of a and h.

The pole ends up on the positive real axis for $h < \frac{2}{a}$.

d. We use $s' = \dfrac{a}{\tan(ah/2)}\dfrac{z-1}{z+1} = \dfrac{a}{\gamma}\dfrac{z-1}{z+1}$ and obtain

$$\begin{aligned} H(z) = G(s') &= \frac{\cancel{a}}{\frac{\cancel{a}}{\gamma}\frac{z-1}{z+1} + \cancel{a}} = \frac{1}{\frac{1}{\gamma}\frac{z-1}{z+1} + 1} = \\ &= \frac{\gamma(z+1)}{(z-1) + \gamma(z+1)} = \frac{\gamma(z+1)}{(1+\gamma)z + \gamma - 1}. \end{aligned}$$

The discrete-time pole is thus located in

$$\bar{z} = \frac{1-\gamma}{1+\gamma} = \frac{1-\tan(ah/2)}{1+\tan(ah/2)} = \frac{1+1-(1+\tan(ah/2))}{1+\tan(ah/2)} = \frac{2}{\tan(ah/2)+1} - 1$$

which is a real number $\forall a \in \mathbb{R}, h > 0$. If we analyze the first term, it is easy to see that it goes from $-\infty$ to ∞ by changing a and/or h within their domains. If we limit the analysis for a while in the range $0 < ah < 3\pi/2$ (since it is the domain of $\bar{z}$), we can see that $\bar{z}$ is monotonically decreasing

$$\frac{d\bar{z}}{d\,ah} = -\frac{\sec^2\left(\frac{ah}{2}\right)}{\left(\tan\left(\frac{ah}{2}\right)+1\right)^2} < 0$$

In addition, we know that for $ah = 0$ then $\bar{z} = 1$, and for $ah = \pi/2$ then $\bar{z} = -\infty$. The limit value for ah is then the one that gives $\bar{z} = -1$, i.e., when $ah = \pi$.

In other words, we can say that the discrete-time system is asymptotically stable whenever $0 < ah < \pi$. Since the value $\bar{z}$ is a periodic function it is easy to extend the result as $2k\pi < ah < (2k+1)\pi$, with $k \in \mathbb{N}$ (recall that $a > 0$ and $h > 0$).

Figure 4.2 illustrates the plot of $\bar{z}$ as a function of ah.

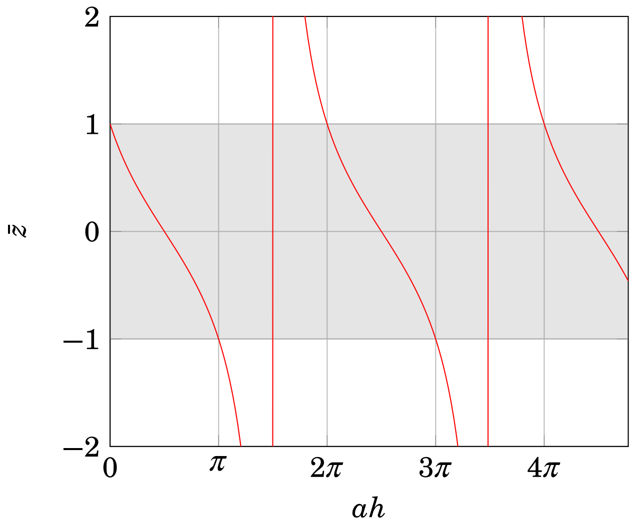

Figure 4.2 Plot of $\bar{z}$ as a function of ah.

Given the analysis above, it is easy to say that the eigenvalue ends up on the positive real axis when $2k\pi < ah < 2k\pi + \pi/2$ with $k \in \mathbb{N}$.

6.1 Consider the task set below.

Task name	T_i	D_i	C_i
A	10	2	1
B	5	4	2
C	20	10	4

a. Assign priorities to the tasks according to the rate-monotonic principle. Draw the schedule assuming worst-case conditions, i.e., the tasks are released simultaneously and the actual execution times are equal to the worst-case execution times. Do the tasks meet their deadlines?

b. Assign priorities to the tasks according to the deadline-monotonic principle. Draw the schedule assuming worst-case conditions. Do the tasks meet their deadlines?

Solution

a. The tasks are given the following priorities:

Task name	T_i	Priority
A	10	Medium
B	5	High
C	20	Low

The schedule is shown in Figure 6.3. The worst-case response times of the tasks are $R_A = 3$, $R_B = 2$, $R_C = 9$, i.e., task A will not meet its deadlines.

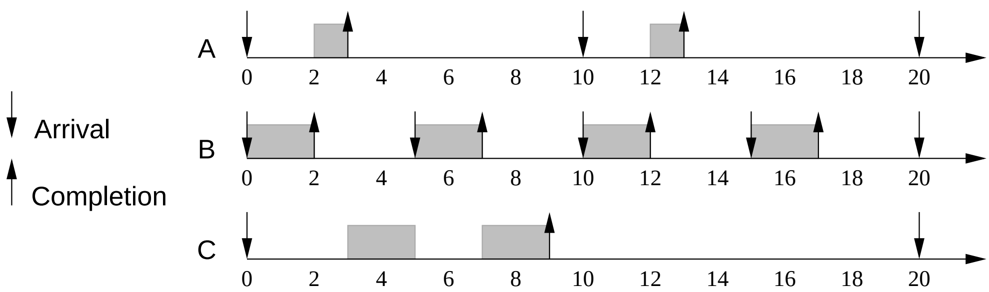

Figure 6.3 Schedule with rate-monotonic priority assignments.

b. The tasks are given the following priorities:

Task name	D_i	Priority
A	2	High
B	4	Medium
C	10	Low

The schedule is shown in Figure 6.4. The worst-case response times of the tasks are $R_A = 1$, $R_B = 3$, $R_C = 9$, i.e. all tasks will meet their deadlines.

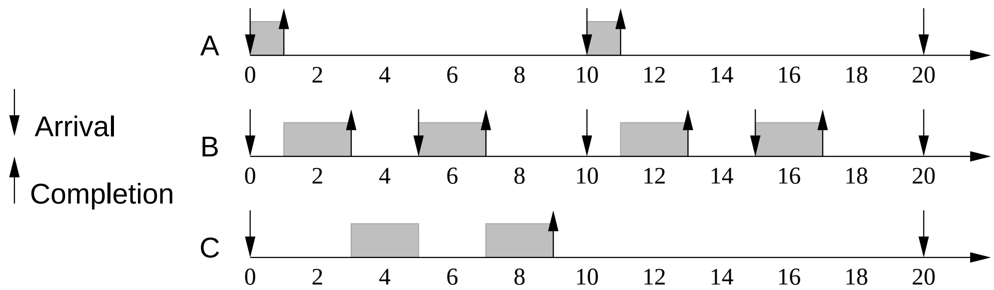

Figure 6.4 Schedule with deadline-monotonic priority assignments.