Systems Engineering/Process Control L3

- Mathematical modeling
- State-space models
- Stability

Reading: *Systems Engineering and Process Control*: 3.1–3.4

Process modeling

- Dynamics in processes often described by differential equations
- Two approaches:
  1. Mathematical modeling
     - Use physical laws (conservation equations etc) to create model
  2. Experiments
     - Create experiments (e.g., step response), analyze input & output
     - FRT041 System identification

In practice, a combination of both methods is often used

Mathematical modeling

- Flow balances
- Intensity balances
- Constitutive relations

Flow balances

- volume flow [m^3/s]

$$\begin{bmatrix}\textit{Change in}\\ \textit{stored volume}\\ \textit{per time unit}\end{bmatrix} = \begin{bmatrix}\textit{Inflow}\end{bmatrix} - \begin{bmatrix}\textit{Outflow}\end{bmatrix}$$

- material flow (mass balance) [mol/s]

$$\begin{bmatrix}\textit{Change in number of}\\ \textit{accumulated particles}\\ \textit{per time unit}\end{bmatrix} = \begin{bmatrix}\textit{Inflow of}\\ \textit{particles}\end{bmatrix} - \begin{bmatrix}\textit{Outflow of}\\ \textit{particles}\end{bmatrix}$$

Flow balances

- energy flow [W]

$$\begin{bmatrix} \textit{Change in} \\ \textit{stored energy} \\ \textit{per time unit} \end{bmatrix} = \begin{bmatrix} \textit{Power in} \end{bmatrix} - \begin{bmatrix} \textit{Power out} \end{bmatrix}$$

- current flow [A]

$$\begin{bmatrix} \textit{Sum current} \\ \textit{to node} \end{bmatrix} = \begin{bmatrix} \textit{Sum current} \\ \textit{from node} \end{bmatrix}$$

Intensity balances

- momentum balance [N]

$$\begin{bmatrix} \textit{Change in} \\ \textit{momentum} \\ \textit{per time unit} \end{bmatrix} = \begin{bmatrix} \textit{Driving} \\ \textit{forces} \end{bmatrix} - \begin{bmatrix} \textit{Braking} \\ \textit{forces} \end{bmatrix}$$

- voltage balance [V]

$$\begin{bmatrix} \textit{Sum voltage around circuit} \end{bmatrix} = 0$$

Constitutional relations

- Ideal gas law

$$p = \frac{nR}{V}T$$

- Torricelli's law

$$v = \sqrt{2gh}$$

- Energy in heated liquid

$$W = C_p \rho V T$$

- Ohm's law

$$u = Ri$$

Typical balance equations for chemical processes

Total mass balance:

$$\frac{d\rho V}{dt} = \sum_{\substack{i=\text{all}\\ \text{inlets}}} \rho_i q_i - \sum_{\substack{i=\text{all}\\ \text{outlets}}} \rho_i q_i$$

Mass balance for component j:

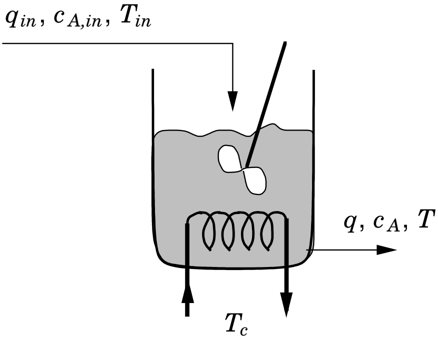
$$\frac{dc_j V}{dt} = \sum_{\substack{i=\text{all}\\ \text{inlets}}} c_{j,i} q_i - \sum_{\substack{i=\text{all}\\ \text{outlets}}} c_{j,i} q_i + r_j V$$

Total energy balance:

$$\frac{dE}{dt} = \sum_{\substack{i=\text{all}\\ \text{inlets}}} \rho_i V_i H_i - \sum_{\substack{i=\text{all}\\ \text{outlets}}} \rho_i V_i H_i + \sum_{\substack{k=\text{all phase}\\ \text{boundaries}}} Q_k + W$$

(all follow from physical conservation laws)

Example: CSTR with exothermic reaction



- Exothermic reaction $A \to B$, $r = k_0 e^{-E/RT} c_A$
- Cooling coil with temperature T_c
- Perfect stirring, constant density ρ

Example: CSTR with exothermic reaction

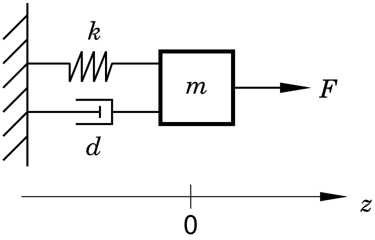
Total mass balance, balance for comp. A, and total energy balance:

$$
\begin{aligned}
\frac{dV}{dt} &= q_{in} - q \\
\frac{dc_A}{dt} &= \frac{q_{in}}{V}(c_{A,in} - c_A) - k_0 e^{-E/RT} c_A \\
\frac{dT}{dt} &= \frac{q_{in}}{V}(T_{in} - T) + \frac{(-\Delta H_r) k_0}{\rho C_p} e^{-E/RT} c_A + \frac{UA}{V \rho C_p}(T_c - T)
\end{aligned}
$$

after simplifications

- Nonlinear third order model
- State variables: V, c_A, T
- Possible inputs: q_{in}, q, $c_{A,in}$, T_{in}, T_c
- Parameters (constants): ρ, C_p, $(-\Delta H_r)$, k_0, E, R, U, A

Example: Mechanical system



- Mass m with position z
- External force: F
- Spring force: $F_k = -kz$
- Damper force: $F_d = -d\dot{z}$

Example: Mechanical system

Momentum balance:

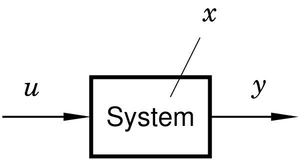
$$m\ddot{z} = F - kz - d\dot{z}$$

Introduce $v = \dot{z} \;\Rightarrow$

$$\begin{aligned}\dot{v} &= -\frac{d}{m}v - \frac{k}{m}z + \frac{1}{m}F\\ \dot{z} &= v\end{aligned}$$

- Linear second order model
- State variables v, z
- Input: F
- Parameters (constants): m, k, d

State-space form



In general, x, u and y are vectors:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$$

- n = number of state variables = system order
- m = number of inputs
- p = number of outputs (measurements)

State-space form

- x is called system state *(state)*
  - it contains values of all accumulated quantities in the system
  - (it represents the system "memory")
- The dynamics are described by n first order differential equations:

$$\frac{dx_1}{dt} = f_1(x_1, \ldots, x_n, u_1, \ldots, u_m)$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_n(x_1, \ldots, x_n, u_1, \ldots, u_m)$$

State-space form

- Outputs described by p algebraic equations (not always stated):

$$\begin{aligned} y_1 &= g_1(x_1, \ldots, x_n, u_1, \ldots, u_m) \\ &\vdots \\ y_p &= g_p(x_1, \ldots, x_n, u_1, \ldots, u_m) \end{aligned}$$

- System can be written in vector form as:

$$\begin{aligned} \frac{dx}{dt} &= f(x, u) && \text{(state equation)} \\ y &= g(x, u) && \text{(measurement equation)} \end{aligned}$$

- (f and g can be nonlinear functions)

State-space form for linear systems

- A system is linear if all f_i and g_i are linear functions
- Example:

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1m}u_m \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m \\
y_1 &= c_{11}x_1 + \ldots + c_{1n}x_n + d_{11}u_1 + \ldots + d_{1m}u_m \\
&\vdots \\
y_p &= c_{p1}x_1 + \ldots + c_{pn}x_n + d_{p1}u_1 + \ldots + d_{pm}u_m
\end{aligned}$$

State space form for linear systems

In matrix form:

$$\frac{dx}{dt} = Ax + Bu \qquad \text{(state equation)}$$
$$y = Cx + Du \qquad \text{(measurement equation)}$$

- x and u are deviations from equilibrium point
- $(x, u) = (0, 0)$ is always in equilibrium (why?)
- D is called system direct term (often 0 for real processes)

Mini problem: What dimensions does matrices A, B, C and D have?

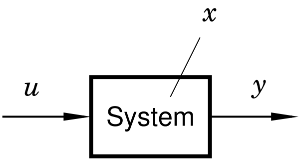
Example: Mechanical system

- State vector: $x = \begin{pmatrix} v \\ z \end{pmatrix}$
- We control $u = F$ and measure $y = z$
- The model on state space form with matrices:

$$\frac{dx}{dt} = \underbrace{\begin{pmatrix} -\frac{d}{m} & -\frac{k}{m} \\ 1 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} \frac{1}{m} \\ 0 \end{pmatrix}}_{B} u$$

$$y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{C} x + \underbrace{\begin{pmatrix} 0 \end{pmatrix}}_{D} u$$

Solution of the linear state equation



- state-space model of system:

$$\frac{dx}{dt} = Ax + Bu \qquad \text{(state equation)}$$
$$y = Cx + Du \qquad \text{(measurement equation)}$$

- How does x (and y) depend on input u and initial state $x(0)$?

Solution of state equation – scalar case

- System with one state variable and one input:

$$\frac{dx(t)}{dt} = ax(t) + bu(t)$$

- Solution:

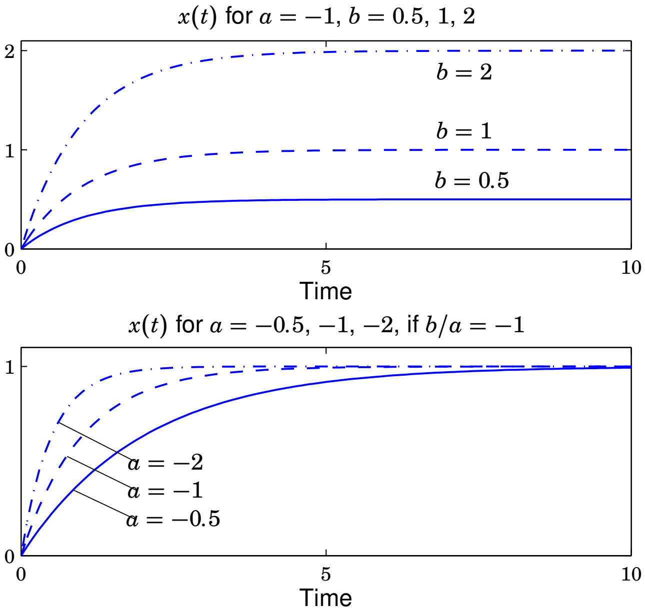
$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau$$

- Example: Constant input $u(t) = u_0$ and $a \neq 0$:

$$x(t) = e^{at}x(0) + \frac{b}{a}(e^{at} - 1)u_0$$

$x(t)$ limited if $a < 0$

Simulation with $u(t) = 1$ and $x(0) = 0$



Solution of state equation – general case

- State space model:

$$\frac{dx}{dt} = Ax + Bu$$

- Solution:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

where e^{At} is matrix exponential function, defined as

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

Example: Mechanical system

- Recall state-space system:

$$\frac{dx}{dt} = \begin{pmatrix} -\frac{d}{m} & -\frac{k}{m} \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} \frac{1}{m} \\ 0 \end{pmatrix} u$$

- Assume:

$$d = 0, \quad F = u = 0, \quad x(0) = \begin{pmatrix} v(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad m = k = 1$$

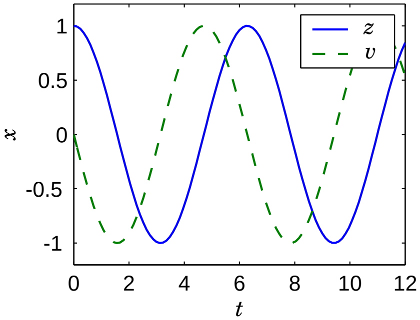
  i.e., no damping, no external force
- Gives state-space system and exponential matrix:

$$\frac{dx}{dt} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} x, \qquad e^{At} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

- Solution:

$$x(t) = e^{At} x(0) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

Simulation of mechanical system



Eigenvalues

- Eigenvalues of A given by n roots to characteristic equation:

$$\det(\lambda I - A) = 0$$

- $\det(\lambda I - A) = P(\lambda)$ is called characteristic polynomial
- Eigenvalues can be complex
- Multiplicity of eigenvalue = nbr of eigenvalues with same value

Eigenvalues

- Suppose A is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_n$:

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- Then

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}$$

- Every eigenvalue λ_i gives a term $e^{\lambda_i t}$ in solution

Eigenvalues

- Assume that A is a general matrix
- Every eigenvalue of A gives a term $P_{m_i-1}(t)e^{\lambda_i t}$ in e^{At} where
  - $P_{m_i-1}(t)$ is a polynomial in t of order at most $m_i - 1$
  - m_i is the multiplicity of the eigenvalue
- Example:
  - A matrix
  $$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$
  - Eigenvalues:
  $$\lambda_1 = \lambda_2 = -1 \quad (m = 2)$$
  - Exponential matrix:
  $$e^{At} = \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

Stability for linear systems

- Stability is a system property – does not depend on input
- Can therefore study the uncontrolled system:

$$\frac{dx}{dt} = Ax$$

- Solution:

$$x(t) = e^{At}x(0)$$

Stability notions

Asymptotic stability: $x(t) \to 0$ as $t \to \infty$ for all initial states

Stability: $x(t)$ limited as $t \to \infty$ for all initial states

Instability: $x(t)$ unlimited as $t \to \infty$ for some initial state

(**Marginally stable**: Stable but not asymptotically stable system)

Example

Asymptotically stable systems:

- Water tank with hole in the bottom
- Temperature in oven
- Speed in a car

(Marginally) Stable systems:

- Water tank without hole in the bottom
- Mass-damper-spring system without damping
- Distance covered in a car

Unstable systems:

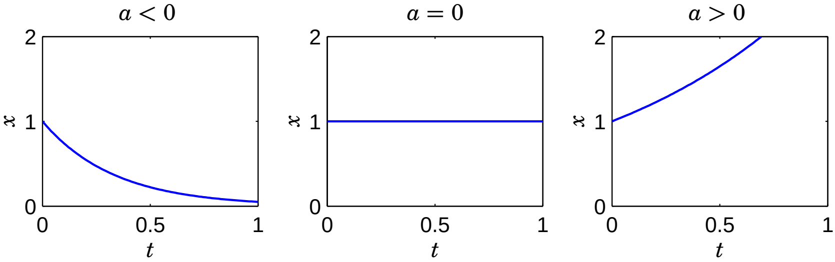
- Inverted pendulum
- Segway

Stability in the scalar case

- State-space model and solution:

$$\frac{dx(t)}{dt} = ax(t), \qquad\qquad x(t) = e^{at}x(0)$$

- Solution plots for different a:



  - asymptotically stable if $a < 0$
  - stable if $a \leq 0$
  - unstable if $a > 0$

Stability in the general case

- Eigenvalues λ_i to A-matrix decides stability
- A linear system is:
  - Asymptotically stable if all $\mathrm{Re}(\lambda_i) < 0$
  - Unstable if some $\mathrm{Re}(\lambda_i) > 0$
  - Stable if all $\mathrm{Re}(\lambda_i) \leq 0$ and possible pure imaginary eigenvalues have multiplicity 1

Routh–Hurwitz stability criteria

Second order systems:

- 2nd order characteristic polynomial (for 2×2-matrix A):

$$\det(\lambda I - A) = P(\lambda) = \lambda^2 + p_1\lambda + p_2$$

- All roots in left half-plane (all $\mathrm{Re}(\lambda_i) < 0$) iff $p_1 > 0$ and $p_2 > 0$

Third order systems:

- 3rd order characteristic polynomial (for 3×3-matrix A):

$$\det(\lambda I - A) = P(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$$

- All roots in left half-plane iff $p_1 > 0$, $p_2 > 0$, $p_3 > 0$ and $p_1 p_2 > p_3$.

Example: Mechanical system

- State-space model:

$$\frac{dx}{dt} = \begin{pmatrix} -\frac{d}{m} & -\frac{k}{m} \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} \frac{1}{m} \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

- Characteristic equation:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda + \frac{d}{m} & \frac{k}{m} \\ -1 & \lambda \end{vmatrix} = \lambda^2 + \frac{d}{m}\lambda + \frac{k}{m} = 0$$

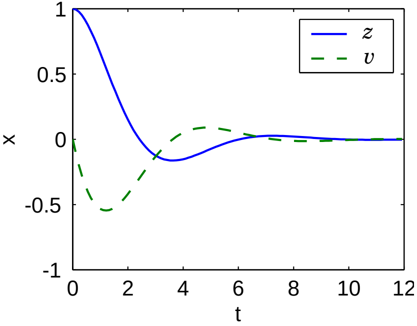
- Suppose $m, k, d > 0$: Asymptotically stable
- Suppose $m, k > 0, d = 0$: Eigenvalues

$$\lambda_{1,2} = \pm i\sqrt{\frac{k}{m}}$$

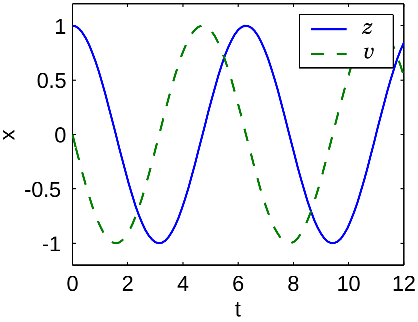
stable (but not asymptotically stable)

Simulation of mechanical system

$m = k = d = 1$:



$m = k = 1, \ d = 0$:



Linear systems on state-space form in MATLAB

```
% Define system matrices
A = [1 2; 3 4];
B = [0; 1];
C = [1 0];
D = 0;

% Create state-space model
sys = ss(A,B,C,D);

% Compute eigenvalues to A matrix
eig(A)

% Simulate system w/o input from initial state x0
initial(sys,x0)
```