Math Repetition for Automatic Control, Basic Course
Solutions

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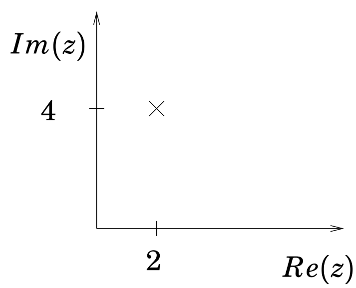
Complex numbers

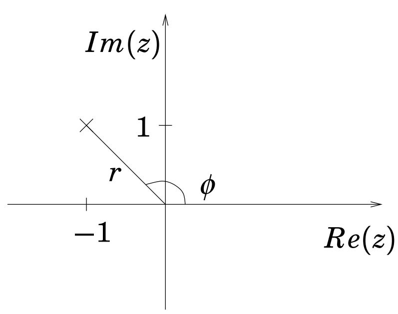
1.

a. $Re(z) = -2$, $Im(z) = 3$. Note that the imaginary part *is not* $3i$.

b. See Figure 1.



Figur 1



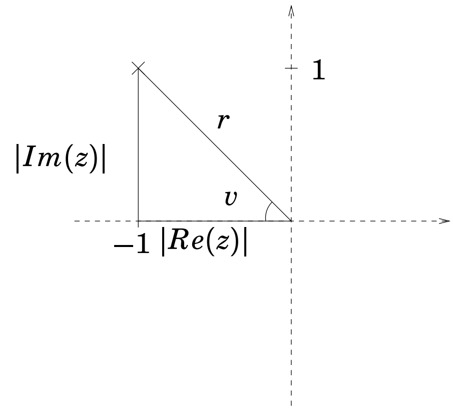
Figur 2

c. See Figur 2. The magnitude $|z| = r$ is the distance to the origin, and the argument $\arg(z) = \phi$ is the angle to the positive real axis.

d. *The magnitude* $|z|$: From Figure 2, we note that Pythagoras' Theorem can be applied:

$$|z| = \sqrt{(Re(z))^2 + (Im(z))^2}$$

This formula can be applied to compute the magnitude of any complex numbers. In our case $Re(z) = -1$ and $Im(z) = 1$. Hence $|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$.



Figur 3

The Argument $\arg(z)$: Here we compute the argument in radians. The angle ϕ i Figure 2 can be computed as $\phi = \pi - v$, where v is the angle in the triangle shown in Figure 3. We have

$$\tan(v) = \frac{Im(z)}{|Re(z)|} \implies v = \arctan\left(\frac{Im(z)}{|Re(z)|}\right)$$

In our case $v = \arctan(1/1) = \arctan 1 = \pi/4$, and thus $\phi = \pi - v = \pi - \pi/4 = 3\pi/4$.

e. A number z can be expressed in polar coordinates as $z = |z|e^{\arg(z)i}$. From the previous problem we know that $|z| = \sqrt{2}$, $\arg(z) = 3\pi/4$. Therefore $z = -1 + i = \sqrt{2}e^{3\pi i/4}$.

f. We can express z as

$$z = 3e^{\pi i} = 3(\cos(\pi) + i\sin(\pi)) = 3(-1 + i \cdot 0) = -3$$

So $Re(z) = -3$, $Im(z) = 0$.

2.

a.

$$|e^{\omega i}| = |\cos(\omega) + i\sin(\omega)| = \sqrt{\cos^2(\omega) + \sin^2(\omega)} = \sqrt{1} = 1$$

Its very useful to know this result by heart.

b. The number $e^{\omega i}$ is a complex number expressed in polar coordinates with magnitude 1 and argument ω. Therefore

$$\arg(e^{\omega i}) = \omega$$

c.

$$|-2(-1+2i)(-4-3i)| = |-2| \cdot |-1+2i| \cdot |-4-3i| =$$
$$2 \cdot \sqrt{(-1)^2 + 2^2} \cdot \sqrt{(-4)^2 + (-3)^2} = 2\sqrt{5}\sqrt{25} = 10\sqrt{5} \approx 22.36$$

d. Arguments are subject to the same rules as logarithms, e.g.

$$\arg(xy^2/z) = \arg(x) + 2\arg(y) - \arg(z)$$

In our case

$$\begin{gathered}\arg(-2(-1+2i)(-4-3i)) = \arg(-2) + \arg(-1+2i) + \arg(-4-3i) = \\ \pi + (\pi + \arctan(2/-1)) + (\pi + \arctan(-3/-4)) = \\ 3\pi + \arctan(-2) + \arctan(3/4) \approx 8.96\end{gathered}$$

e.

$$\left|\frac{2e^{-5i}(2-i)^2}{2i+3}\right| = \frac{2|e^{-5i}||2-i|^2}{|2i+3|} = \frac{2\cdot 1(2^2+(-1)^2)}{\sqrt{2^2+3^2}} = \frac{10}{\sqrt{13}} \approx 2.77$$

f.

$$\begin{gathered}\arg\left(\frac{2e^{-5i}(2-i)^2}{2i+3}\right) = \arg(2) + \arg(e^{-5i}) + 2\arg(2-i) - \arg(2i+3) = \\ 0 + (-5) + 2\arctan(-1/2) - \arctan(2/3) \approx -3.51\end{gathered}$$

Second order polynomial equations

3. The solution to $x^2 + px + q = 0$, where p and q are constants, is given by

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

In our case $p = -1$, $q = 4$, and thus

$$x_{1,2} = -\frac{-1}{2} \pm \sqrt{\left(\frac{-1}{2}\right)^2 - 4} = \frac{1}{2} \pm \sqrt{-\frac{15}{4}} = \frac{1}{2} \pm i\frac{\sqrt{15}}{2} \approx 0.5 \pm 1.94i$$

4. In order to use the formula above, we divide both sides by 3.

$$x^2 + \frac{2}{3}x + \frac{1}{3} = 0$$

According to the formula ($p = 2/3$, $q = 1/3$) we have

$$x_{1,2} = -\frac{1}{3} \pm \sqrt{\frac{1}{9} - \frac{1}{3}} = -\frac{1}{3} \pm i\frac{\sqrt{2}}{3} \approx -0.33 \pm 0.47i$$

Partial fractions expansion

5. Suppose that $f(x)$ can be expressed as

$$f(x) = \frac{1}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2}$$

We have

$$f(x) = \frac{a}{x+1} + \frac{b}{x+2} = \frac{a(x+2) + b(x+1)}{(x+1)(x+2)} = \frac{x(a+b) + 2a + b}{(x+1)(x+2)}$$

By identification of the parameters, we obtain the following system of equations

$$a + b = 0$$
$$2a + b = 1$$

The solution is $a = 1$ and $b = -1$, and therefore

$$f(x) = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

6. Proceeding as we did above, we have:

$$\begin{aligned} f(x) &= \frac{3x+11}{(x+1)(x-3)(x+2)} = \frac{a}{x+1} + \frac{b}{x-3} + \frac{c}{x+2} \\ &= \frac{a(x-3)(x+2) + b(x+1)(x+2) + c(x+1)(x-3)}{(x+1)(x-3)(x+2)} \\ &= \frac{x^2(a+b+c) + x(-a+3b-2c) - 6a + 2b - 3c}{(x+1)(x-3)(x+2)} \end{aligned}$$

We now need to solve

$$a + b + c = 0$$
$$-a + 3b - 2c = 3$$
$$-6a + 2b - 3c = 11$$

The solution is $a = -2$, $b = 1$, $c = 1$, and therefore

$$f(x) = \frac{3+11x}{(x+1)(x-3)(x+2)} = -\frac{2}{x+1} + \frac{1}{x-3} + \frac{1}{x+2}$$

7. Start by determining the roots of the polynomial in the denominator

$$x^2 + 3x + 2 = 0 \implies x_1 = -1, \quad x_2 = -2$$

We can express $f(x)$ as

$$f(x) = \frac{2}{x^2 + 3x + 2} = \frac{2}{(x+1)(x+2)}$$

Proceeding as we did above results in

$$\begin{aligned} f(x) &= \frac{2}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2} = \frac{a(x+2) + b(x+1)}{(x+1)(x+2)} \\ &= \frac{x(a+b) + 2a + b}{(x+1)(x+2)} \end{aligned}$$

The solution to

$$\begin{aligned} a + b &= 0 \\ 2a + b &= 2 \end{aligned}$$

is $a = 2$, $b = -2$. Therefore

$$f(x) = \frac{2}{x^2 + 3x + 2} = \frac{2}{x+1} - \frac{2}{x+2}$$

Matrices

8.

a.

$$A \cdot B = \begin{pmatrix} -1 \cdot 1 + 0 \cdot 4 & -1 \cdot -2 + 0 \cdot -5 \\ 3 \cdot 1 + 2 \cdot 4 & 3 \cdot -2 + 2 \cdot -5 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 11 & -16 \end{pmatrix}$$

b.

$$A \cdot B = \begin{pmatrix} -1 \cdot 1 & -1 \cdot 2 \\ 3 \cdot 1 & 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 3 & 6 \end{pmatrix}$$

c.

$$A \cdot B = \begin{pmatrix} -1 \cdot 4 + 0 \cdot -5 \end{pmatrix} = -4$$

9.

$$\det(A) = -2 \cdot 0 - 4 \cdot 1 = -4$$

The formula for determining the determinant of a 2×2 matrix can be found in the formula sheet.

10.

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

The formula for inverting a 2×2 matrix can be found in the formula sheet.

11.

a. The eigenvalus λ of a matrix A satisfy the following equation (this equation is also part of the formula sheet)

$$\det(\lambda I - A) = 0$$

In our case

$$\det(\lambda I - A) = \det\left(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) = \det\left(\begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{pmatrix}\right)$$
$$= (\lambda - 1)(\lambda - 4) - (-2) \cdot (-3) = \lambda^2 - 5\lambda - 2 = 0$$

By solving this second order polynomial equation, we obtain

$$\lambda = \frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 + 2} \implies \lambda_1 = -0.37,\ \lambda_2 = 5.37$$

b. Here A is diagonal and the eigenvalues are given by the diagonal elements, $\lambda_1 = -1$, $\lambda_2 = 4$, $\lambda_3 = -2$.

12.

a. The system of equations can be expressed as

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 3 \\ -1 \end{pmatrix} x_2 = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

which is the same as

$$\begin{pmatrix} 5 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

b. The system of equations can be expressed as

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Taylor series expansion

13.

a. A function $f(x)$ can be exapnded in a Taylor series around a point a. I.e. $f(x)$ can be expressed as

$$f(x) = f(a) + \frac{1}{1!}\frac{df}{dx}(a)(x-a) + \frac{1}{2!}\frac{d^2 f}{dx^2}(x-a)^2 + \ldots$$

We can then obtain an approximation of $f(x)$ around $x = a$ by only keeping some of the first few terms. The approximation is good provided that x stays sufficiently close to a.

Our task was to expand $f(x)$ up to first order terms, i.e. the two first terms in the Taylor series. We want to expand $f(x)$ around the point $x = 2$, i.e. $a = 2$.

We have

$$f(x) \approx f(2) + \frac{df}{dx}(2)(x-2)$$

where

$$f(2) = 4, \quad \frac{df}{dx} = 2x, \quad \frac{df}{dx}(2) = 4$$

The result is

$$f(x) \approx 4 + 4(x-2) = 4(x-1)$$

b. Here $f(x,u)$ is a function of two varaibles and the Taylor series expansion at $x = a$, $u = b$ is given by

$$f(x,u) = f(a,b) + \frac{1}{1!}\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{1}{1!}\frac{\partial f}{\partial u}(a,b)(u-b) +$$
$$+\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial x \partial u}(a,b)(x-a)(u-b) + \frac{1}{2!}\frac{\partial^2 f}{\partial u^2}(a,b)(u-b)^2 + \ldots$$

Since our task is to expand up to first order terms, we keep the constant $f(a,b)$, and all terms that contain first order derivatives of $f(x,u)$.

In our case we have

$$f(x,u) \approx f(3,\pi) + \frac{\partial f}{\partial x}(3,\pi)(x-3) + \frac{\partial f}{\partial u}(3,\pi)(u-\pi)$$

where

$$f(3,\pi) = 15 - 0 = 15, \quad \frac{\partial f}{\partial x} = 5\sqrt{3}\frac{1}{2}x^{-\frac{1}{2}} = \frac{5}{2}\sqrt{\frac{3}{x}}, \quad \frac{\partial f}{\partial x}(3,\pi) = \frac{5}{2} = 2.5,$$
$$\frac{\partial f}{\partial u} = \cos(u) \quad \frac{\partial f}{\partial u}(3,\pi) = -1$$

The result is

$$f(x,u) \approx 15 + 2.5(x-3) - 1(u-\pi) \approx 10.64 - 2.5x - u$$