# AUTOMATIC CONTROL

# Exercises

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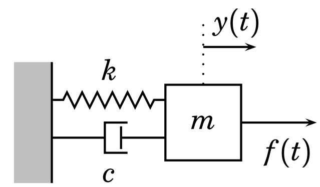
# 1. Model Building and Linearization

**1.1** In the right figure, a mass $m$ is attached to a wall with a spring and a damper. The spring has a spring constant $k$ and the damper has a damping constant $c$. It is assumed that $k > c^2/4m$. An external force $f$ is acting on the mass. We denote the translation of the mass from its equilibrium position by $y$. Further, we let $f(t)$ be the input signal and $y(t)$ be the output signal. The force equation gives

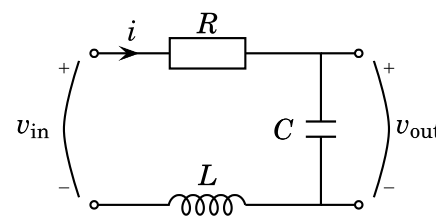



$$m\ddot{y} = -ky - c\dot{y} + f$$

**a.** Introduce the states $x_1 = y$ and $x_2 = \dot{y}$ and write down the state space representation of the system.

**b.** Assume that the system is at rest at $t = 0$ and that $f(t)$ changes from 0 to 1 as a step at $t = 0$. What is the resulting $y(t)$? Sketch the solution.

**1.2** In the RLC circuit to the right, the input and output voltages are given by $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$, respectively. By means of Kirchhoff's voltage law we see that



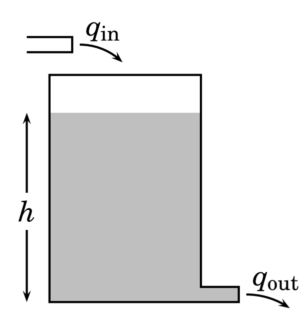

$$v_{\text{in}} - Ri - v_{\text{out}} - L\frac{di}{dt} = 0$$

For the capacitor, we additionally have

$$C\dot{v}_{\text{out}} = i$$

Introduce the states $x_1 = v_{\text{out}}$ and $x_2 = \dot{v}_{\text{out}}$ and give the state space representation of the system.

**1.3** A cylindrical water tank with cross section $A$ has an inflow $q_{\text{in}}$ and an outflow $q_{\text{out}}$. The outlet area is $a$. Under the assumption that the outlet area is small in comparison to the cross section of the tank, Torricelli's law $v_{\text{out}} = \sqrt{2gh}$ is valid and gives the outflow rate.




**a.** What would be a suitable state variable for this system? Determine a differential equation, which tells how the state variable depends on the inflow $q_{\text{in}}$.

**b.** Give the state space representation of the dependence of the inflow $q_{in}$ on the outflow $q_{out}$.

**c.** Let the inflow be constant, $q_{\text{in}} = q^0_{\text{in}}$. Determine the corresponding constant tank level $h^0$ and outflow $q^0_{\text{out}}$. Linearize the system around this stationary point.

**d.** Assume that the system is at equilibrium with inflow $q_{\text{in}} = q^0_{\text{in}}$. If the inflow is suddenly turned off, what will the outflow $q_{\text{out}}(t)$ become according to the nonlinear and linear equations, respectively?

**1.4** Give the state space representation of the system

$$\dddot{y} + 3\ddot{y} + 2\dot{y} + y = u$$

where $u(t)$ and $y(t)$ are the input and output, respectively. Choose states $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = \ddot{y}$.

**1.5** A process with output $y(t)$ and input $u(t)$ is described by the differential equation

$$\ddot{y} + \sqrt{y} + y\dot{y} = u^2$$

**a.** Introduce states $x_1 = y$, $x_2 = \dot{y}$ and give the state space representation of the system.

**b.** Find all stationary points $(x_1^0, x_2^0, u^0)$ of the system.

**c.** Linearize the system around the stationary point corresponding to $u^0 = 1$.

**1.6** Linearize the system

$$\begin{aligned} \dot{x}_1 &= x_1^2 x_2 + \sqrt{2}\sin u & &(= f_1(x_1, x_2, u)) \\ \dot{x}_2 &= x_1 x_2^2 + \sqrt{2}\cos u & &(= f_2(x_1, x_2, u)) \\ y &= \arctan \frac{x_2}{x_1} + 2u^2 & &(= g(x_1, x_2, u)) \end{aligned}$$

around the stationary point $u^0 = \pi/4$.

**1.7** For a process with input $u(t)$ and output $y(t)$ it holds that

$$\ddot{y} + (1 + y^4)\dot{y} = \sqrt{u + 1} - 2$$

**a.** Write the differential equation in state space form.

**b.** Linearize the state space equations around the point $u^0 = 3$, $y^0 = 1$, $\dot{y}^0 = 0$.

**1.8** A simple model of a satellite, orbiting the earth, is given by the differential equation

$$\ddot{r}(t) = r(t)\omega^2 - \frac{\beta}{r^2(t)} + u(t)$$

where $r$ is the satellite's distance to the earth and $\omega$ is its angular acceleration, see figure 1.1. The satellite has an engine, which can exert a radial force $u$.

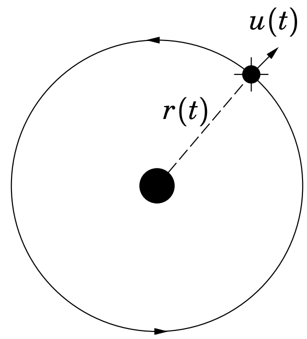



**Figure 1.1** Satellite orbiting the earth.

**a.** Introduce the state vector

$$x(t) = \begin{pmatrix} r(t) \\ \dot{r}(t) \end{pmatrix}$$

and write down the nonlinear state space equations for the system.

**b.** Linearize the state space equations around the stationary point

$$(r, \dot{r}, u) = (r^0, 0, 0)$$

Consider $r$ as the output and give the state space representation of the linear system. Express $r^0$ in $\beta$ and $\omega$.

# 2. Dynamical Systems

**2.1** Determine the transfer functions and give differential equations, describing the relation between input and output for the following systems, respectively.

**a.**

$$\dot{x} = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 5 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} -1 & 1 \end{pmatrix} x + 2u$$

**b.**

$$\dot{x} = \begin{pmatrix} -7 & 2 \\ -15 & 4 \end{pmatrix} x + \begin{pmatrix} 3 \\ 8 \end{pmatrix} u$$
$$y = \begin{pmatrix} -2 & 1 \end{pmatrix} x$$

**c.**

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} x + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x + 5u$$

**d.**

$$\dot{x} = \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 2 \end{pmatrix} x + 3u$$

**2.2** Determine the impulse and step responses of the systems in assignment 2.1.

**2.3** Derive the formula $G(s) = C(sI - A)^{-1}B + D$ for a general system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

**2.4** Consider the system

$$G(s) = \frac{1}{s^2 + 4s + 3}$$

**a.** Calculate the poles and zeros of the system.

**b.** What is the static gain of the system?

**c.** Calculate and sketch the step response of the system.

**2.5** Consider the system

$$G(s) = \frac{0.25}{s^2 + 0.6s + 0.25}$$

**a.** Calculate the poles and zeros of the system.

**b.** What is the static gain of the system?

**c.** Calculate and sketch the step response of the system.

**2.6** Determine the transfer function and poles of the oscillating mass in assignment 1.1. Explain how the poles move if one changes $k$ and $c$, respectively. Can the poles end up in the right half plane?

**2.7** Determine the transfer function of

**a.** the RLC circuit in assignment 1.2,

**b.** the linearized tank in assignment 1.3.

**2.8** Sketch the step response of the processes with the following transfer functions

**a.** $G(s) = \dfrac{2}{s + 2/3}$

**b.** $G(s) = \dfrac{8}{s^2 + s + 4}$

**c.** $G(s) = \dfrac{s^2 + 6s + 8}{s^2 + 4s + 3}$

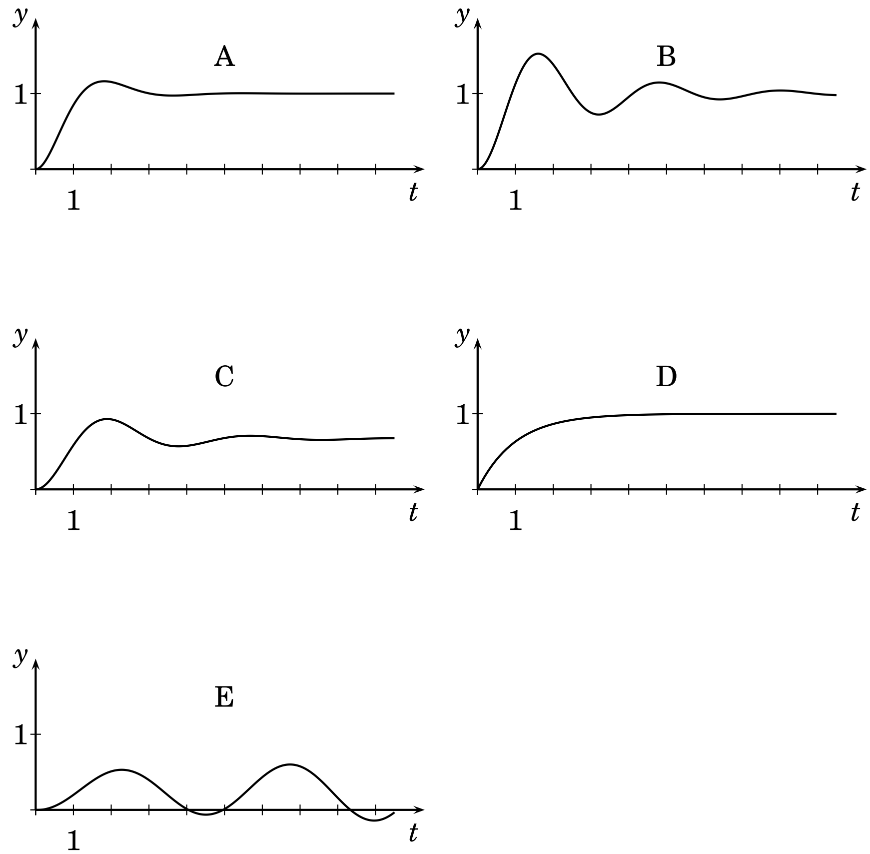
**2.9** Determine which five of the following transfer functions correspond to the step responses A–E below.

$$G_1(s) = \frac{0.1}{s + 0.1} \qquad G_2(s) = \frac{4}{s^2 + 2s + 4}$$

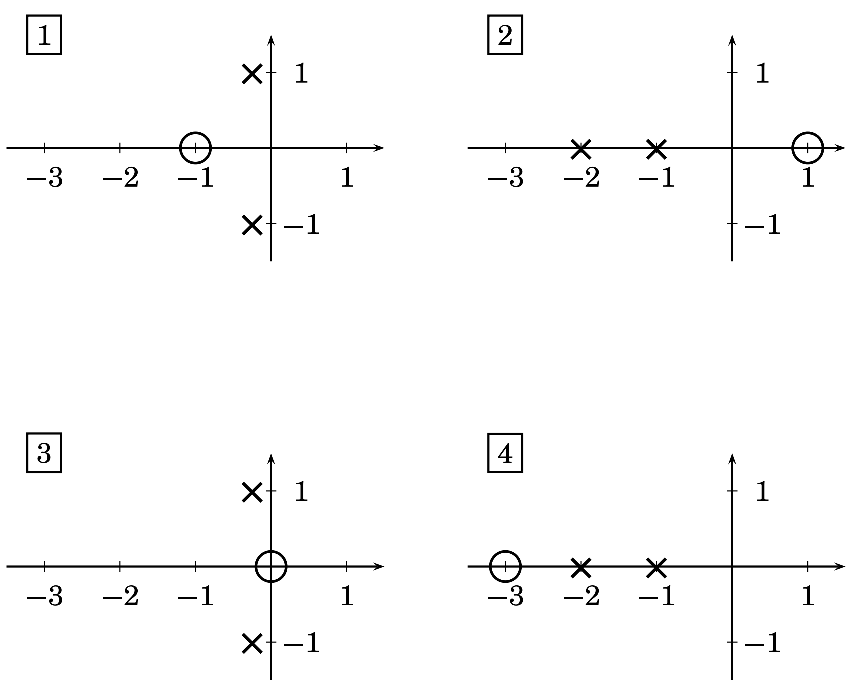
$$G_3(s) = \frac{0.5}{s^2 - 0.1s + 2} \qquad G_4(s) = \frac{-0.5}{s^2 + 0.1s + 2}$$

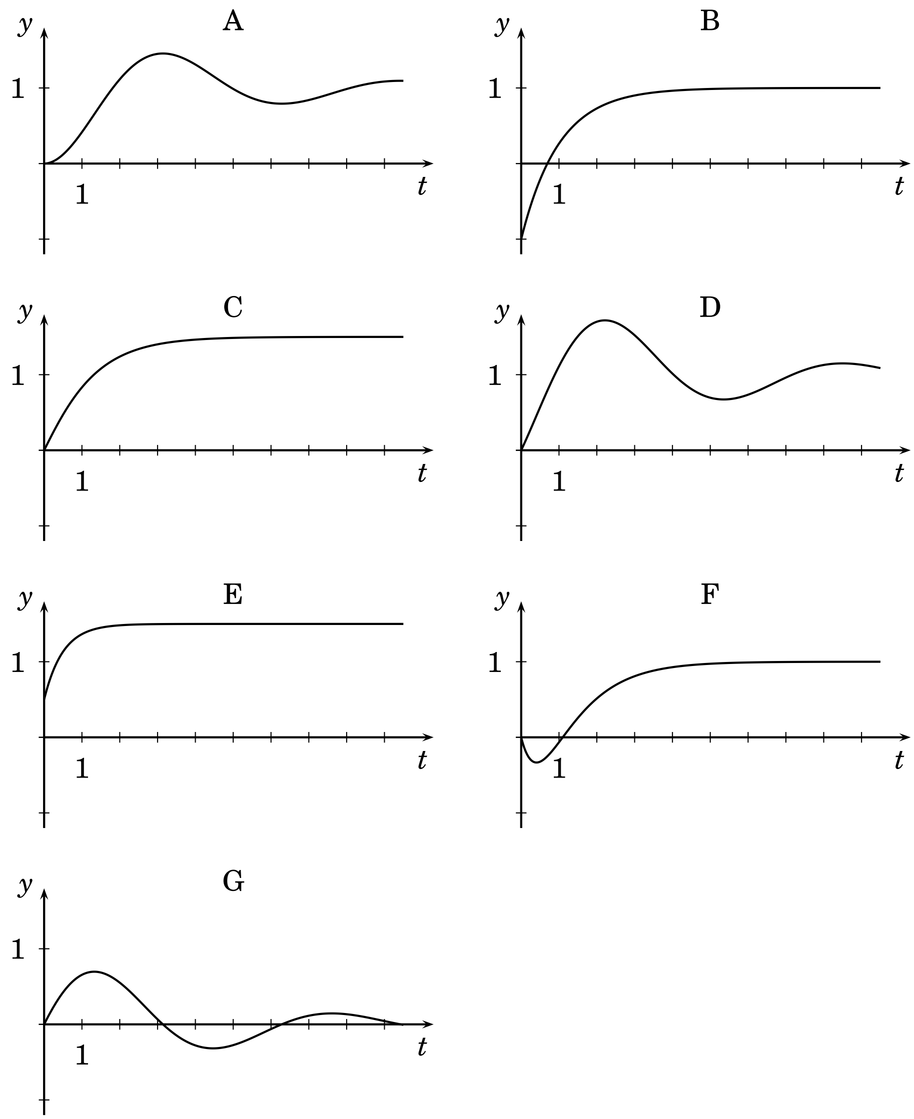
$$G_5(s) = \frac{1}{s + 1} \qquad G_6(s) = \frac{4}{s^2 + 0.8s + 4}$$

$$G_7(s) = \frac{2}{s^2 + s + 3}$$



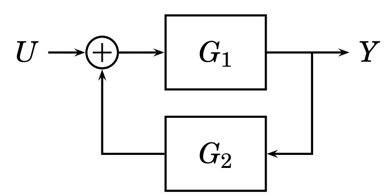

**2.10** Pair each of the four pole-zero plots with the corresponding step responses A–G.



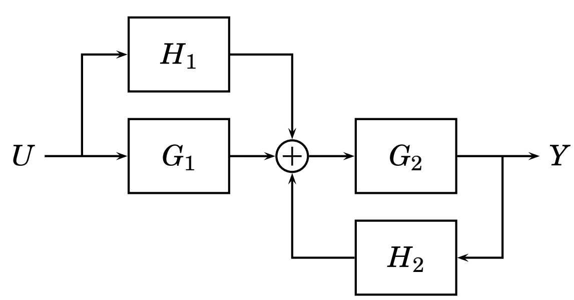



**2.11** Determine the transfer function from $U$ to $Y$ for the systems below.

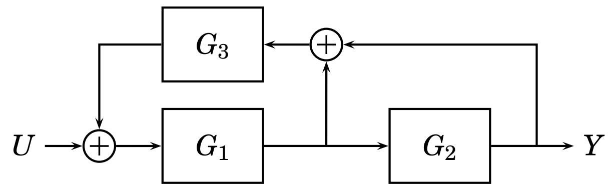
**a.**



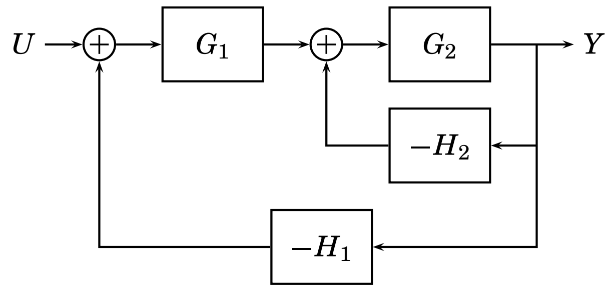

**b.**



**c.**




**d.**




**2.12** Consider the transfer function

$$G(s) = \frac{s^2 + 6s + 7}{s^2 + 5s + 6}$$

Write the system in

**a.** diagonal form,

**b.** controllable canonical form,

**c.** and observable canonical form.

# 3. Frequency Analysis

**3.1** Assume that the system

$$G(s) = \frac{0.01(1 + 10s)}{(1 + s)(1 + 0.1s)}$$

is subject to the input $u(t) = \sin 3t, \quad -\infty < t < \infty$

**a.** Determine the output $y(t)$.

**b.** The Bode plot of the system is shown in figure 3.1. Determine the output $y(t)$ by using the Bode plot instead.

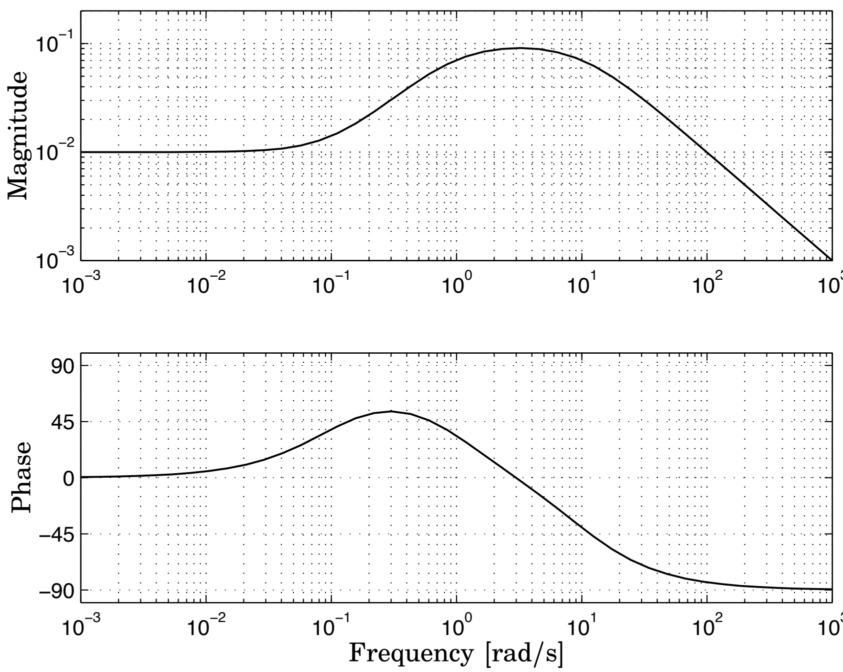



**Figure 3.1** The Bode plot in assignment 3.1.

**3.2** Assume that the oscillating mass in assignment 1.1 has $m = 0.1$ kg, $c = 0.05$ Ns/cm and $k = 0.1$ N/cm. The transfer function is then given by

$$G(s) = \frac{10}{s^2 + 0.5s + 1}$$

**a.** Let the mass be subject to the force $f = \sin \omega t, -\infty < t < \infty$. Calculate the output for $\omega = 0.2$, 1 and 30 rad/s.

**b.** Instead, use the Bode plot of the system in figure 3.2 to determine the output for $\omega = 0.2$, 1 and 30 rad/s.

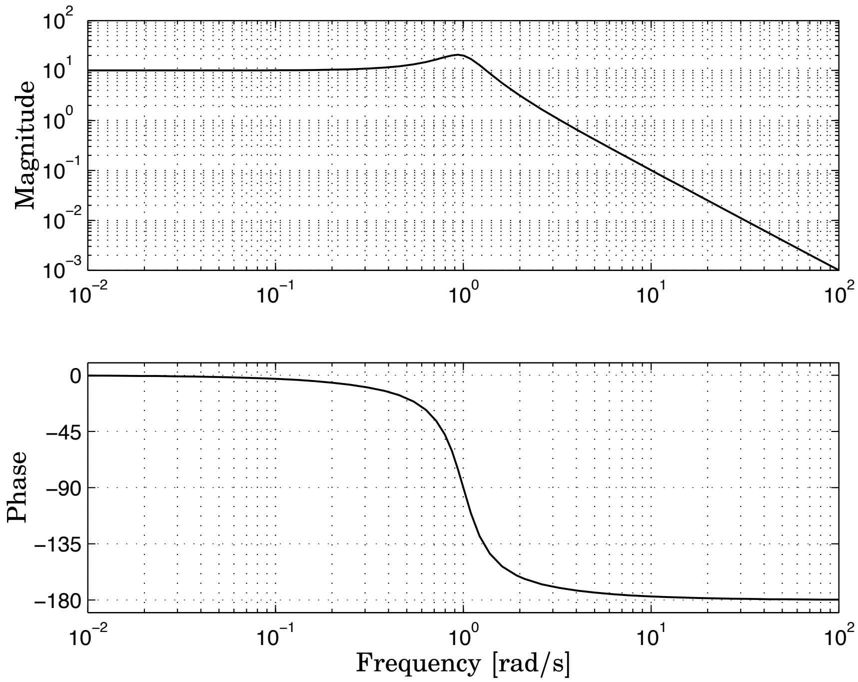



**Figure 3.2** The Bode plot of the oscillating mass in assignment 3.2.

**3.3** Draw the Bode plots corresponding to the following transfer functions

**a.**
$$G(s) = \frac{3}{1 + s/10}$$

**b.**
$$G(s) = \frac{10}{(1 + 10s)(1 + s)}$$

**c.**
$$G(s) = \frac{e^{-s}}{1 + s}$$

**d.**
$$G(s) = \frac{1 + s}{s(1 + s/10)}$$

**e.**
$$G(s) = \frac{2(1 + 5s)}{s(1 + 0.2s + 0.25s^2)}$$

**3.4** Exploit the results from the previous assignment in order to draw the Nyquist curves of

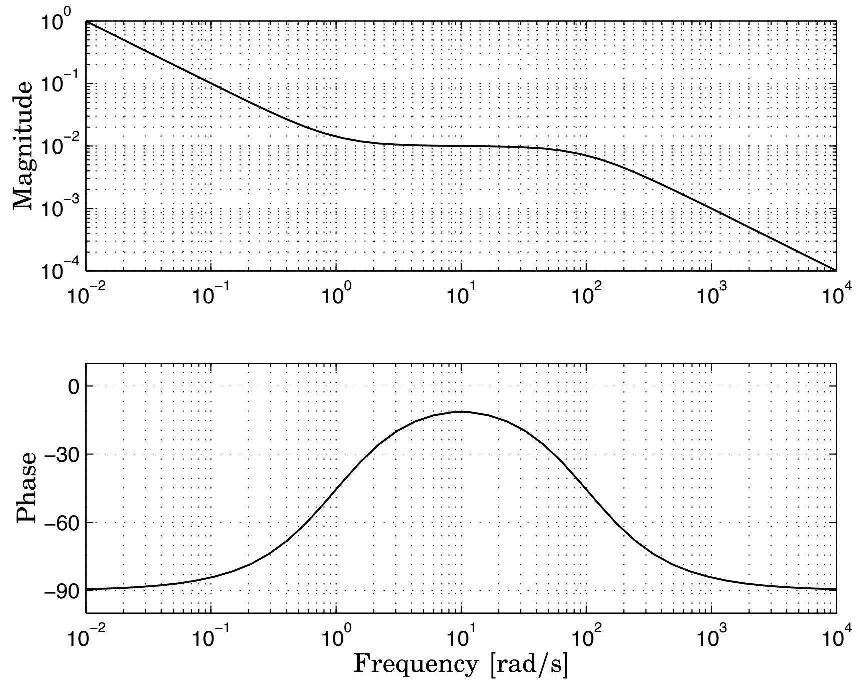
**a.**
$$G(s) = \frac{3}{1 + s/10}$$

**b.**
$$G(s) = \frac{10}{(1 + 10s)(1 + s)}$$

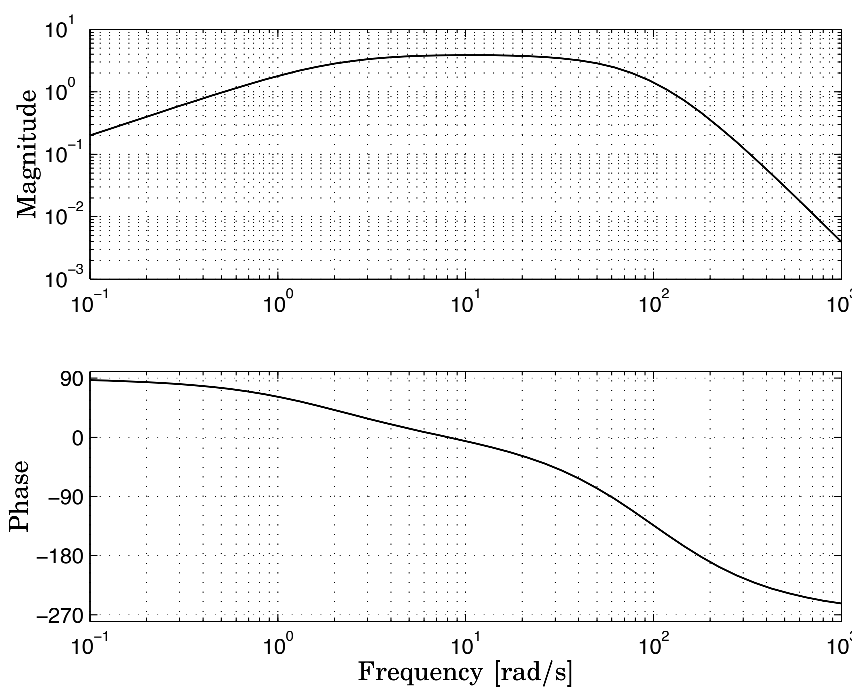
**c.**

$$G(s) = \frac{e^{-s}}{1+s}$$

**3.5** The Bode plot below was obtained by means of frequency response experiments, in order to analyze the dynamics of a stable system. What is the transfer function of the system?




**3.6** Measurements resulting in the Bode plot below have been conducted in order to analyze the dynamics of an unknown system. Use the Bode plot to determine the transfer function of the system. Assume that the system is stable and lacks complex poles and zeros.



# 4. Feedback Systems

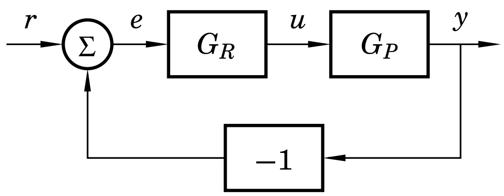
**4.1** Assume that the air temperature $y$ inside an oven is described by the differential equation

$$\dot{y}(t) + 0.01y(t) = 0.01u(t)$$

where $u$ is the temperature of the heating element.

**a.** Let $u$ be the input and $y$ the output and determine the transfer function $G_P(s)$ of the oven.

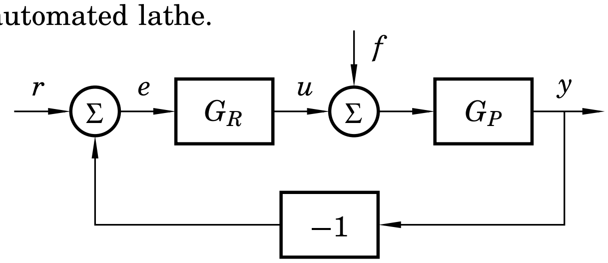
**b.** The oven is to be controlled by a P controller, $G_R(s) = K$, according to the block diagram below. Write down the transfer function of the closed loop system.




**c.** Choose $K$ such that the closed loop system obtains the characteristic polynomial

$$s + 0.1$$

**4.2** The below figure shows a block diagram of a hydraulic servo system in an automated lathe.




The measurement signal $y(t)$ represents the position of the tool head. The reference tool position is $r(t)$, and the shear force is denoted $f(t)$. $G_R$ is the transfer function of the position sensor and signal amplifier, while $G_P$ represents the dynamics of the tool mount and hydraulic piston

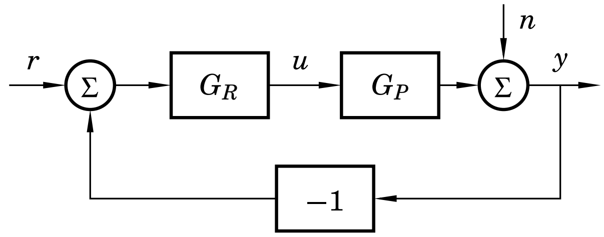
$$G_P(s) = \frac{1}{ms^2 + ds}$$

Here $m$ is the mass of the piston and tool mount, whereas $d$ is the viscous damping of the tool mount. In the assignment it is assumed that $r(t) = 0$.

**a.** How large does the deviation $e(t) = r(t) - y(t)$ between the reference- and measured tool head position become in stationarity if the shear force $f(t)$ is a unit step? The controller is assumed to have a constant gain $G_R(s) = K$.

**b.** How is this error changed if the amplifier is replaced by a PI controller with transfer function $G_R(s) = K_1 + K_2/s$?

**4.3** A process is controlled by a P controller according to the figure below.

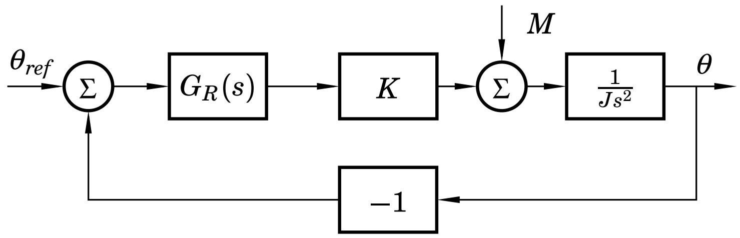



**a.** Measurements of the process output indicate a disturbance $n$. Calculate the transfer functions from $n$ to $y$ and $n$ to $u$.

**b.** Let $G_P(s) = \frac{1}{s+1}$ and assume that the disturbance consists of a sinusoid $n(t) = A \sin \omega t$. What will $u$ and $y$ become, after the decay of transients?

**c.** Assume that $K = 1$ and $A = 1$ in the previous sub-assignment. Calculate the amplitude of oscillation in $u$ and $y$ for the cases $\omega = 0.1$ and 10 rad/s, respectively.

**4.4** The below figure shows a block diagram of a gyro stabilized platform. It is controlled by an motor which exerts a momentum on the platform. The angular position of the platform is sensed by a gyroscope, which outputs a signal proportional to the platform's deviation from the reference value. The measurement signal is amplified by an amplifier with transfer function $G_R$.




It is desired that step changes in the reference $\theta_{ref}$ or the disturbance momentum $M$ on the platform do not result in persisting angular errors. Give the *form* of the transfer function $G_R$, which guarantees that the above criteria hold. Hint: Postulate $G_R(s) = Q(s)/P(s)$

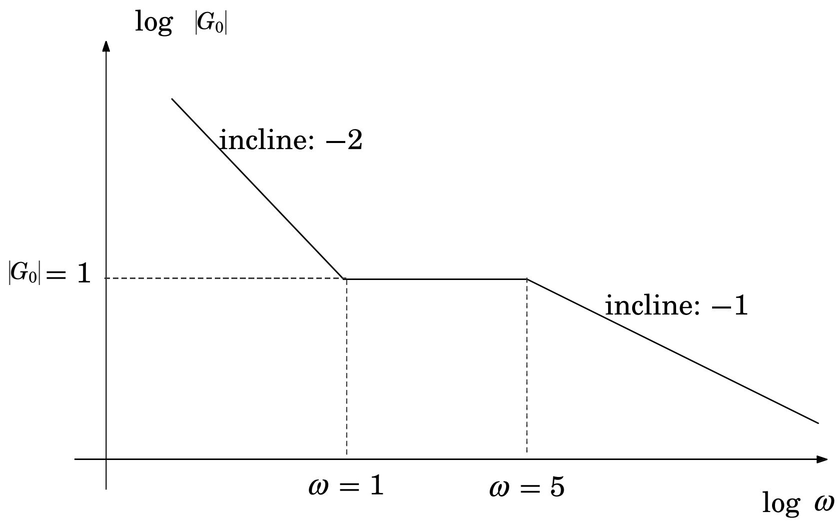
**4.5** When heating a thermal bath, one can assume that the temperature increases linearly with 1°C/s. The temperature is measured by means of a thermocouple with transfer function

$$G(s) = \frac{1}{1 + sT}$$

with time constant $T = 10$ s.

After some initial oscillations, a stationary state, in the sense that the temperature measurement increases with constant rate, is reached. At a time instant, the temperature measurement reads 102.6°C. Calculate the actual temperature of the bath.

**4.6** Consider the system $G_0(s)$ with the following asymptotic gain curve. Assume that the system lacks delays and right half plane zeros.




Further assume that the system is subject to negative feedback and that the closed loop system is stable. Which of the following inputs can be tracked by the closed loop system, without a stationary error? Assume that $r(t) = 0$ for $t < 0$, and that the constants $a$, $b$ and $c \neq 0$.

**a.** $r(t) = a$

**b.** $r(t) = bt$

**c.** $r(t) = ct^2$

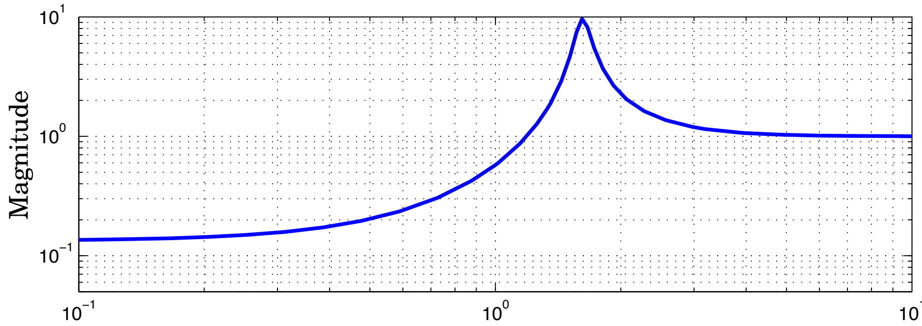
**d.** $r(t) = a + bt$

**e.** $r(t) = \sin(t)$

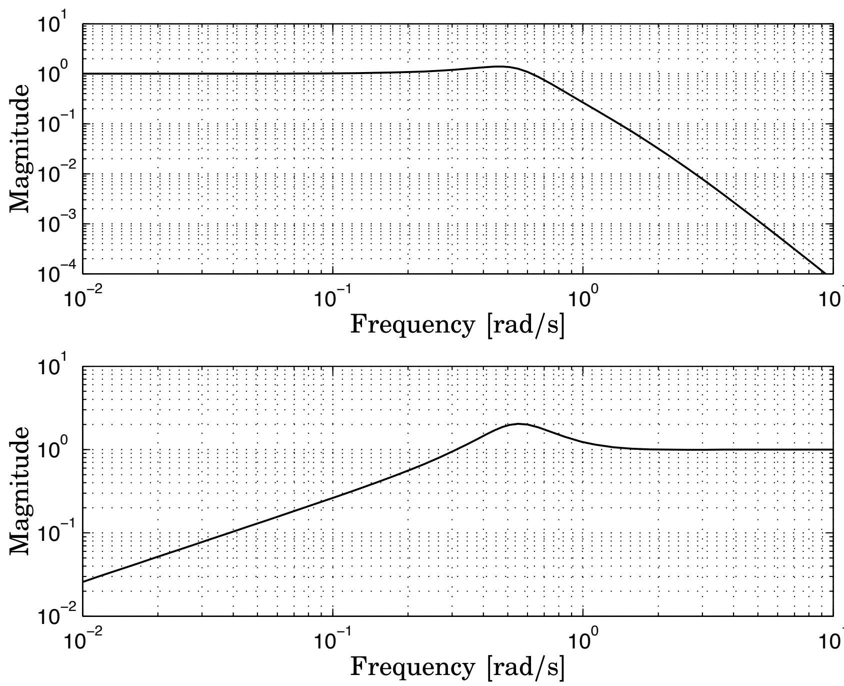
**4.7** In a simple control circuit, the process and controller are given by $G_P(s) = \dfrac{1}{(s+1)^3}$ and $G_R(s) = 6.5$, respectively.

**a.** Determine the sensitivity function $S(s)$.

**b.** The gain plot of the sensitivity function is given below. How much are constant load disturbances damped by the control circuit (in closed loop, as compared to open loop)? At which angular frequency does the control circuit exhibit the largest sensitivity towards disturbances and by how much are disturbances amplified at most?




**4.8** The below figure shows the gain curves of the sensitivity function $S$ and complementary sensitivity function $T$ for a normal control circuit.




**a.** Determine which curve corresponds to the sensitivity function and complementary sensitivity function, respectively.

**b.** Give the frequency range where disturbances are amplified by the feedback loop, and the frequency range where they are damped by the feedback loop. What is the maximum gain of disturbance amplification?

**c.** Give the frequency ranges where the output exhibits good tracking of the reference signal.

**d.** What is the minimal distance between the Nyquist curve of the open loop system and the point $-1$ in the complex plane? What does this say about the gain margin?

# 5. Stability

**5.1** Consider the linear time invariant system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & -1 \end{pmatrix} x$$

**a.** Is the system asymptotically stable?

**b.** Is the system stable?

**5.2** In a simple control loop, the open loop transfer function is given by

$$G_o(s) = G_R(s)G_P(s) = \frac{K}{s(s+2)}$$

Draw the root locus of the characteristic equation of the closed loop system, with respect to the gain parameter $K$.

**5.3** A simple control loop has the open loop transfer function

$$G_o(s) = G_R(s)G_P(s) = \frac{K(s+10)(s+11)}{s(s+1)(s+2)}$$

**a.** Which values of $K$ yield a stable closed loop system?

**b.** Sketch the characteristics of the root locus.

**5.4** Does the transfer function

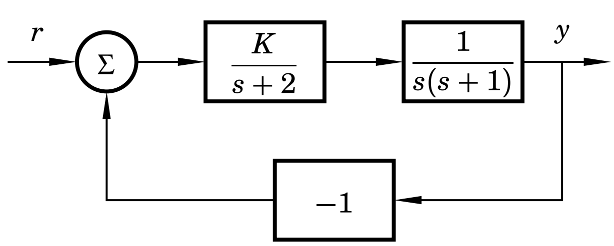
$$G(s) = \frac{s+4}{s^3 + 2s^2 + 3s + 7}$$

have any poles in the right half plane?

**5.5** The figure below shows the block diagram of a printer.

**a.** Which values of the gain $K$ yield an asymptotically stable system?

**b.** The goal is to track a reference which increases linearly with rate 0.1 V/s, and guarantee a stationary error of less than 5 mV. Can this be achieved by adequate tuning of the gain $K$?



**5.6** The open loop transfer function of a simple control loop is given by

$$G_o(s) = G_R(s)G_P(s) = \frac{K}{s(s+1)(s+2)}$$

Use Cauchy's argument principle and the Nyquist theorem in order to find the gains $K$ that result in a stable closed loop system.

**5.7** Consider the Nyquist curves in figure 5.1. Assume that the corresponding systems are controlled by the P controller

$$u = K(r - y)$$

In all cases the open loop systems lack poles in the right half plane. Which values of $K$ yield a stable closed loop system?

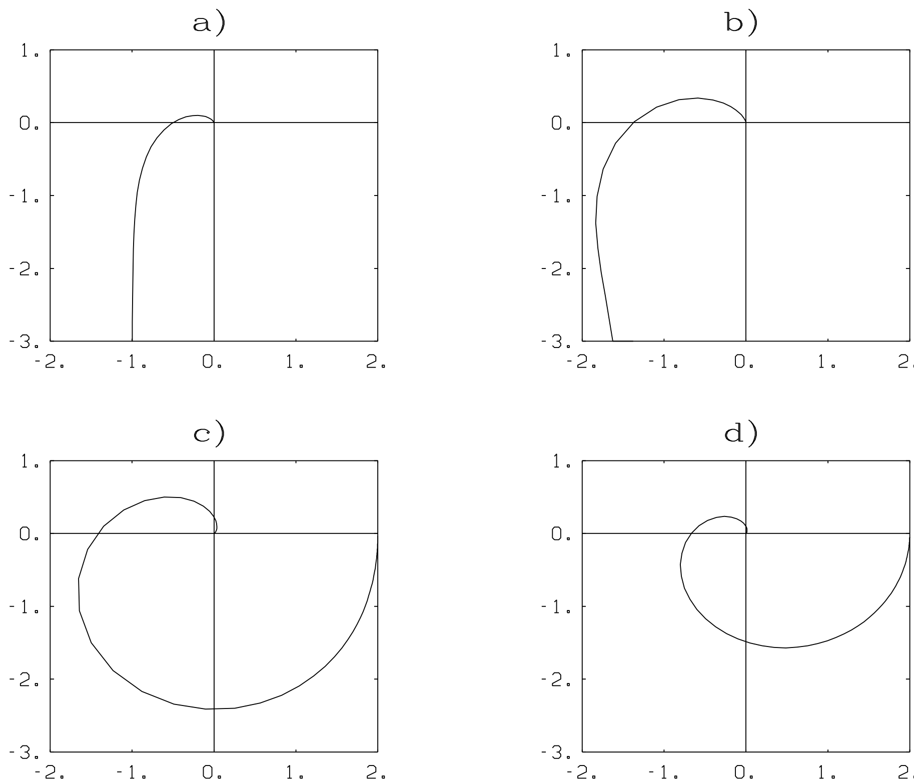



**Figure 5.1** Nyquist curves in assignment 5.7.

**5.8** The transfer function of a process is given by

$$G_p(s) = \frac{1}{(s+1)^3}$$

The loop is closed through proportional feedback

$$u = K(r - y)$$

Use the Nyquist criterion to find the critical value of the gain $K$ (i.e. the value for which the system transits from stability to instability).

**5.9** The Nyquist curve of a system is given in figure 5.2. The system is stable, i.e. lacks poles in the right half plane.

Assume that the system is subject to proportional feedback

$$u = K(r - y)$$

Which values of the gain $K$ result in a stable closed loop system.

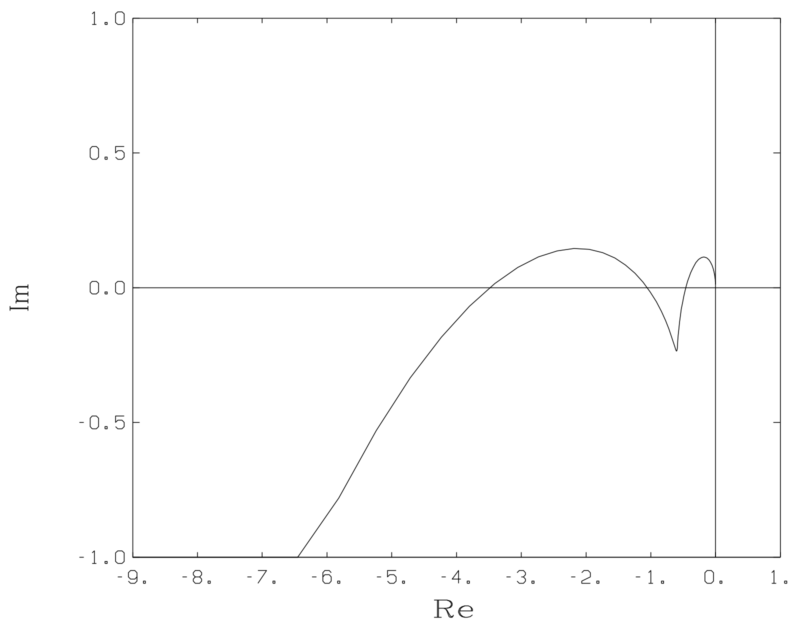



**Figure 5.2** Nyquist curve of the system in assignment 5.9.

**5.10** In order to obtain even product quality in a cement oven, it is crucial that the burn zone temperature is held constant. This is achieved by measuring the burn zone temperature and controlling the fuel flow with a proportional controller. A block diagram of the system is shown in figure 5.3.

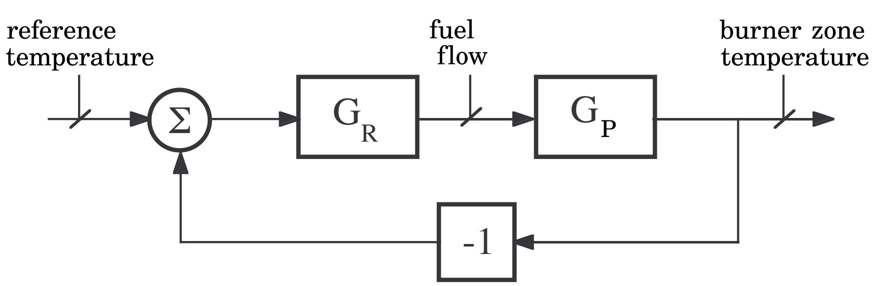



**Figure 5.3** Block diagram of the cement oven with temperature controller in assignment 5.10.

Find the maximal value of the controller gain $K$, such that the closed loop system remains stable? The transfer function from fuel flow to burn zone temperature is given by

$$G_P(s) = \frac{e^{-9s}}{(1+20s)^2}$$

**5.11** In a distillation column, the transfer function from supplied energy to liquid phase concentration of a volatile component is

$$G_P(s) = \frac{e^{-sL}}{1+10s}$$

where time is measured in minutes. The process is controlled by a PI controller with transfer function

$$G_R(s) = 10\left(1 + \frac{1}{2s}\right)$$

What is the maximal permitted transportation delay $L$, yielding at least a 10° phase margin?

**5.12** A process with transfer function $G_P(s)$ is subject to feedback according to figure 5.4. All poles of $G_P(s)$ lie in the left half plane and

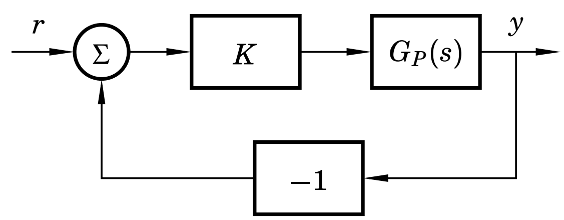



**Figure 5.4** The closed loop system in assignment 5.12.

the Nyquist curve of $G_P$ is shown in figure 5.5. It is assumed that $\arg G_P(i\omega)$ is decreasing and that $G_P(s)$ has more poles than zeros. Further, it holds that the closed loop system is stable for $K = 1$.

Which of the below alternatives are true? Motivate!

**a.** The gain margin $A_m < 2$ for $K = 1$.

**b.** The phase margin $\varphi_m < 45°$ for $K = 1$.

**c.** The phase margin decreases with decreasing gain $K$.

**d.** For $K = 2$ the closed loop system becomes unstable.

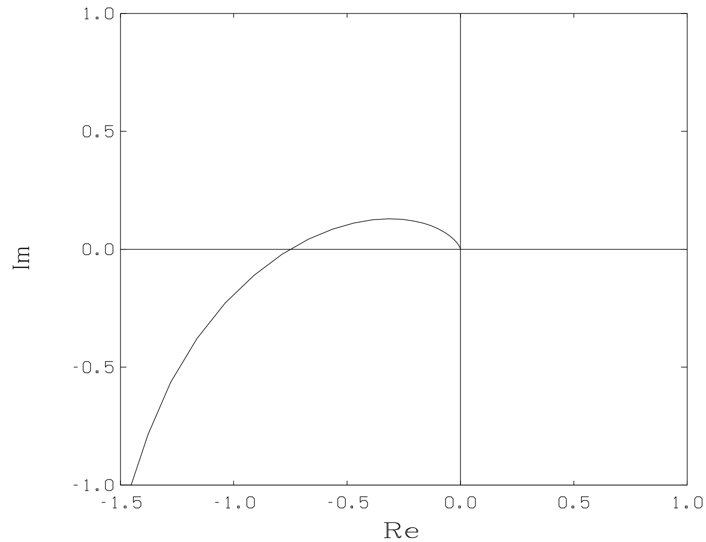



**Figure 5.5** Nyquist curve of the process $G_P(s)$ in assignment 5.12.

**5.13** The Bode plot of the open loop transfer function, $G_o = G_R G_P$, is shown in Figure 5.6. Assume that the system is subject to negative feedback.

**a.** How much can the the gain of the controller or process be increased without making the closed loop system unstable?

**b.** How much additional negative phase shift can be introduced at the cross-over frequency without making the closed loop system unstable?

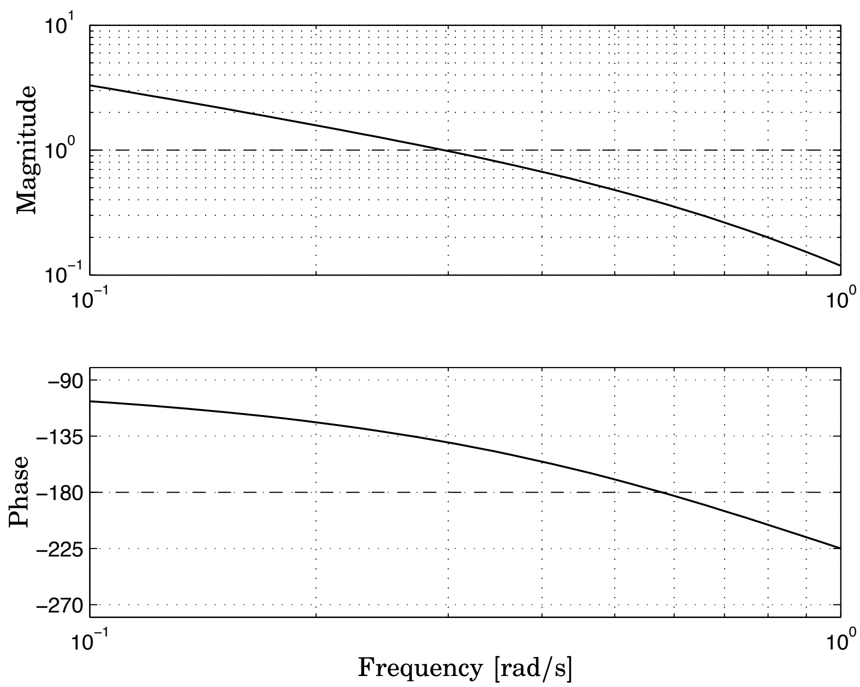



**Figure 5.6** Bode plot of the open loop system in figure 5.13.

**5.14** A Bode plot of the open loop transfer function of the controlled lower tank in the double tank process is shown in figure 5.7. What is the delay margin of the system?

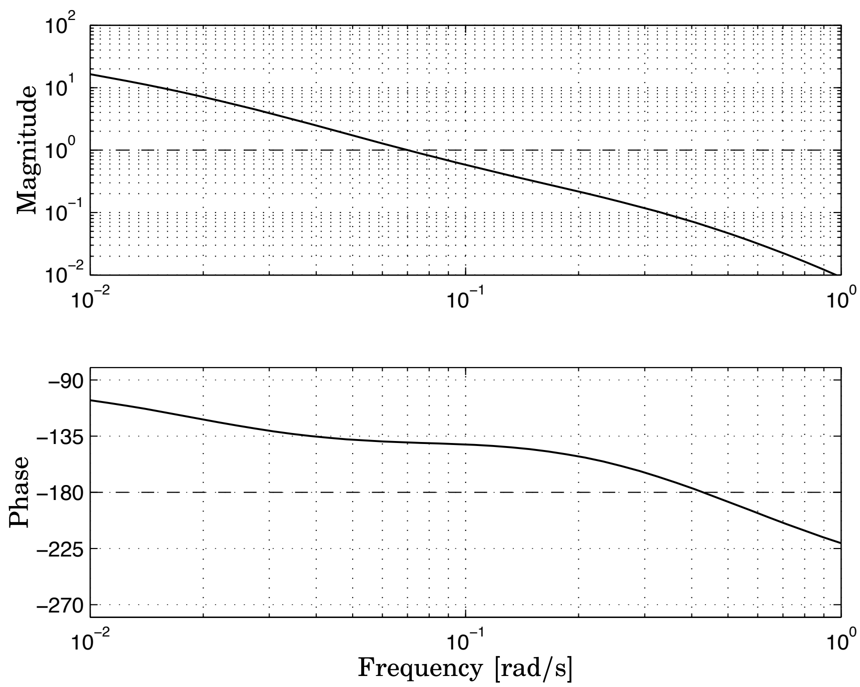



**Figure 5.7** Bode plot of the open loop transfer function of the controlled lower tank in the double tank process in problem 5.14.

# 6. Controllability and Observability

**6.1** A linear system is described by the matrices

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} \beta \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & \gamma \end{pmatrix} \qquad D = 0$$

**a.** For which values of $\beta$ is the system controllable?

**b.** For which values of $\gamma$ is the system observable?

**6.2** A linear system is described by the matrices

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 0 & -1 \end{pmatrix}$$

Find the set of controllable states.

**6.3** Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

Is it observable? If not, find the set of unobservable states.

**6.4** Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u, \qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Which of the states $(3 \quad 0.5)^T$, $(5 \quad 5)^T$, $(0 \quad 0)^T$, $(10 \quad 0.1)^T$ or $(1 \quad -0.5)^T$ can be reached in finite time?

**6.5** Consider the following system with two inputs

$$\frac{dx}{dt} = \begin{pmatrix} -2 & 3 \\ 1 & -4 \end{pmatrix} x + \begin{pmatrix} 1 & 8 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$y = \begin{pmatrix} 3 & 7 \end{pmatrix} x$$

**a.** Is it controllable?

**b.** Assume that we only have authority over $u_1$. Is the system controllable in this case?

**c.** Assume that the two inputs are coupled, so that $u_1 + 2u_2 = 0$. Is the system controllable in this case?

**6.6** Consider the following system

$$\frac{dx}{dt} = \begin{pmatrix} -1 & 1 & 0 \\ -5 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} x$$

**a.** Is the system controllable? Find the set of controllable states.

**b.** Is the system observable? Find the set of unobservable states.

**6.7** A dynamic system is described by the state space model below

$$\dot{x} = \begin{pmatrix} -2 & 2 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 5 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

**a.** Is the system controllable? Which states can be reached in finite time from the initial state $x(0) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$?

**b.** Calculate the transfer function of the system.

**c.** Can the same input-output relation be described with fewer states? Write down such a representation, if possible.

# 7. PID Control

**7.1** A PID controller has the transfer function

$$G_R(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right)$$

**a.** Determine the gain and phase shift of the controller at an arbitrary frequency $\omega$.

**b.** At which frequency does the controller have its minimal gain? What is the gain and phase shift for this frequency?

**7.2** The process

$$G(s) = \frac{1}{(s+1)^3}$$

is controlled by a PID controller with $K = 2$, $T_i = 2$ and $T_d = 0.5$. In order to investigate the effect of changing the PID parameters, we will change $K$, $T_I$ and $T_d$ by a certain factor, one at a time. We will observe how this affects both the step response (from reference and load disturbance) and the Bode plot of the controlled open loop system.

The reference is a unit step at $t = 0$ whereas the load disturbance is a negative unit step.

**a.** We start by studying what happens when the parameters are quadrupled, one at a time. Figure 7.1 shows the nominal case $(K, T_i, T_d) = (2, 2, 0.5)$ (solid curves) together with the cases $(8, 2, 0.5)$, $(2, 8, 0.5)$ and $(2, 2, 2)$. Pair the three Bode plots and the step responses of figure 7.1 with the three cases.

**b.** We now study what happens when each parameter is decreased by a factor 2. The nominal case $(K, T_i, T_d) = (2, 2, 0.5)$ (solid curves) is shown in figure 7.2 together with the cases $(1, 2, 0.5)$, $(2, 1, 0.5)$ and $(2, 2, 0.25)$.

Pair the three Bode plots and the three step responses in figure 7.2 with these three cases.

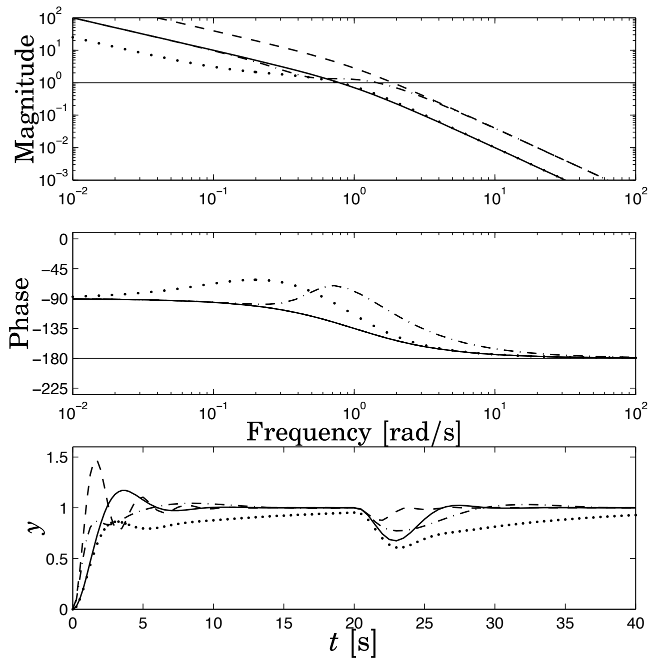



**Figure 7.1** Bode plot and step response for the case when the PID parameters in sub-assignment 7.2b have been multiplied by four. The solid curves correspond to the nominal case.

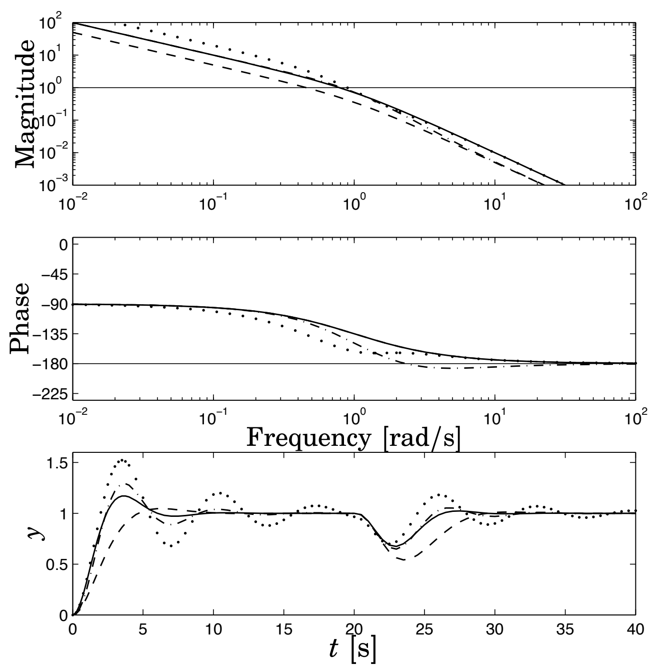



**Figure 7.2** Bode plot and step response for the case when the PID parameters in sub-assignment 7.2b have been divided by two. The solid curves correspond to the nominal case.

**7.3** The steer dynamics of a ship are approximately described by

$$J\frac{dr}{dt} + Dr = C\delta$$

where $r$ is the yaw rate [rad/s] and $\delta$ is the rudder angle [rad]. Further, $J$ [kgm$^2$] is the momentum of inertia wrt the yaw axis of the boat, $D$ [Nms] is the damping constant and $C$ [Nm/rad] is a constant describing the rudder efficiency. Let the rudder angle $\delta$ be the control signal. Give a PI controller for control of the yaw rate, such that the closed loop system obtains the characteristic equation

$$s^2 + 2\zeta\omega s + \omega^2 = 0$$

**7.4** An electric motor can approximately be described by the differential equation

$$J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} = k_i I$$

where $J$ is the moment of inertia, $D$ is a damping constant and $k_i$ is the current constant of the motor. Further, $\theta$ denotes the turning angle and $I$ the current through the motor. Let $\theta$ be the measurement signal and $I$ the control signal. Determine the parameters of a PID controller such that the closed loop system obtains the characteristic equation

$$(s + a)(s^2 + 2\zeta\omega s + \omega^2) = 0$$

Discuss how the parameters of the controller depend on the desired specifications on $a$, $\zeta$ and $\omega$.

**7.5 a.** Draw the Bode plot of a PI controller (let $K = 1$ and $T_i = 1$).

**b.** Draw the Bode plot of a PD controller (let $K = 1$ and $T_d = 15$).

**7.6** A cement oven consists of a long, inclined, rotating cylinder. Sediment is supplied into its upper end and clinkers emerge from its lower end. The cylinder is heated from beneath by an oil burner. It is essential that the combustion zone temperature is kept constant, in order to obtain an even product quality. This is achieved by measuring the combustion zone temperature and controlling the fuel flow with a PI controller. A block diagram of the system is shown in figure 7.3.

The transfer function from fuel flow to combustion zone temperature is given by

$$G_P(s) = \frac{e^{-9s}}{(1 + 20s)^2}$$

and the transfer function of the controller is

$$G_R(s) = K(1 + \frac{1}{sT_i})$$

Use Ziegler-Nichol's frequency method to determine the parameters of the controller.

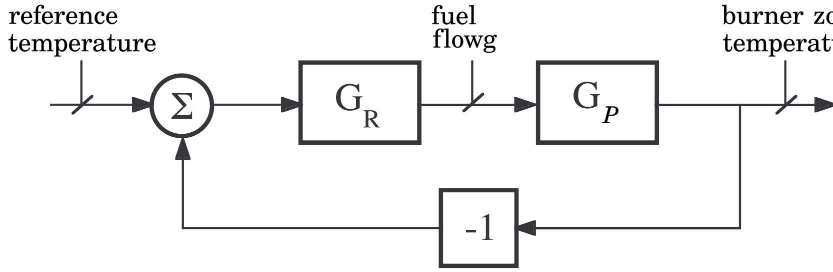



**Figure 7.3** Block diagram of a cement oven with temperature controller.

**7.7** Use Ziegler-Nichol's step response and frequency method, to determine the parameters of a PID controller for a system with the step response and Nyquist curve given in figure 7.4.

**7.8** Consider a system with the transfer function

$$G(s) = \frac{1}{s+1}e^{-s}$$

**a.** Draw the step response of the system and use Ziegler-Nichol's step response method to determine the parameters of a PID controller. Write down the values of the obtained controller parameters $K$, $T_i$ and $T_d$.

**b.** Use Ziegler-Nichol's frequency method to determine the parameters of a PID controller. Compare it to the controller which was obtained using Ziegler-Nichol's step response method in sub-assignment a.

**7.9** Consider a system with transfer function

$$G(s) = \frac{1}{(s+1)^3}$$

Calculate the parameters $K$, $T_i$ and $T_d$ of the PID controller, by applying Ziegler-Nichol's frequency method.

**7.10** A process is to be controlled by a PID controller obtained through Ziegler-Nichol's methods.

**a.** Use the step response method for the process with the solid step response curve in figure 7.5.

**b.** The Nyquist curve of the same system is shown in figure 7.6.

The point marked 'o' corresponds to the frequency $\omega = 0.429$ rad/s. Apply the frequency method to the process.

**c.** Unfortunately the step response method results in an unstable closed loop system. The frequency method yields a stable but poorly damped system. The reason why the step response method works so badly,

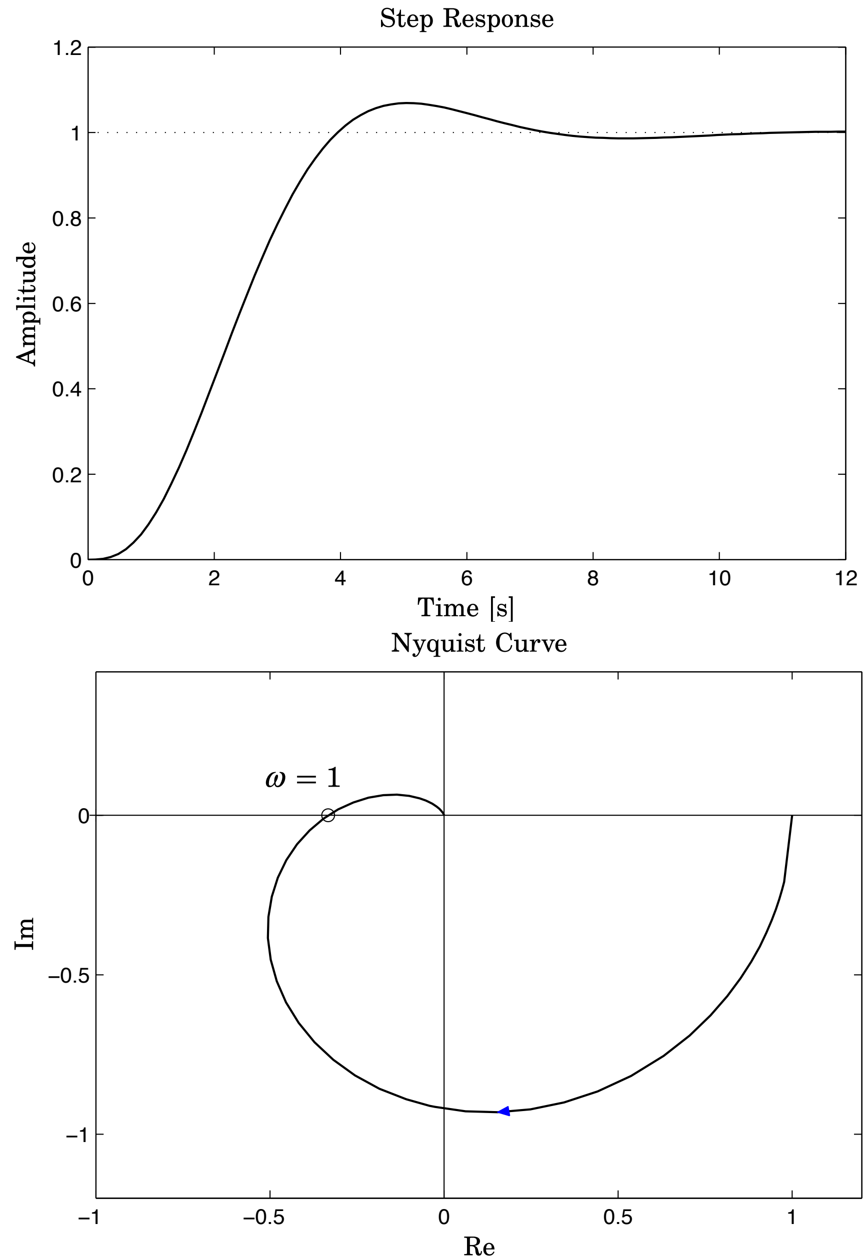



**Figure 7.4** Step response and Nyquist curve of the system in assignment 7.7.

is that it tries to approximate the process with a delayed first order system (the dashed step response above).

By exploiting the Nyquist curve, one can obtain PID parameters yielding the solid curve step response in figure 7.7. The dashed and dotted curves were obtained through the step response method.

How do you think $K$ has changed in the third method, as compared to the Ziegler-Nichol's methods (increase or decrease)?

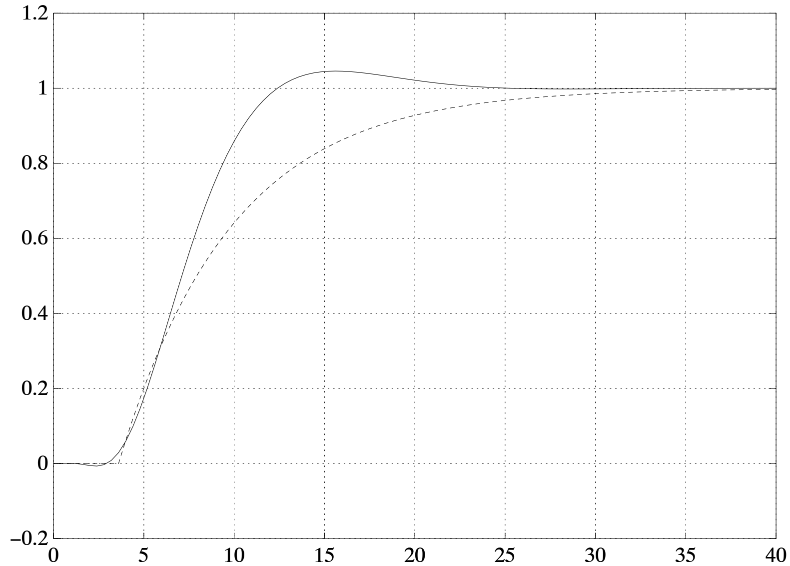



**Figure 7.5**

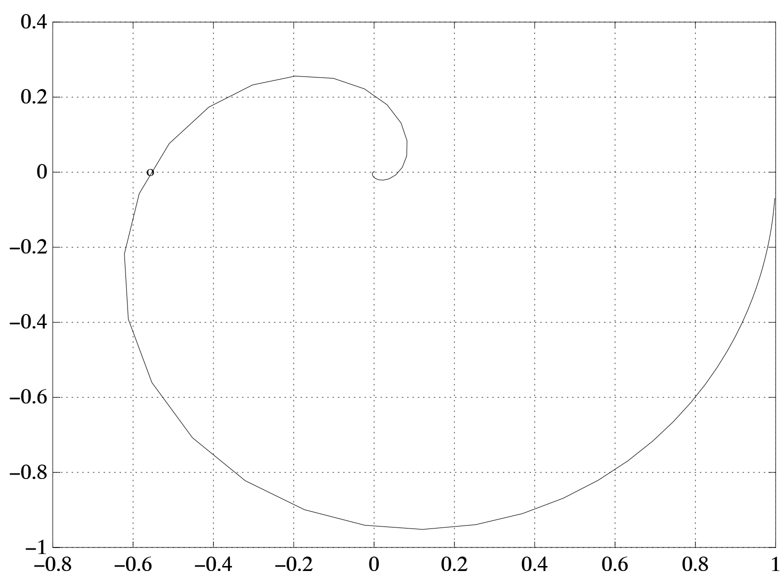



**Figure 7.6**

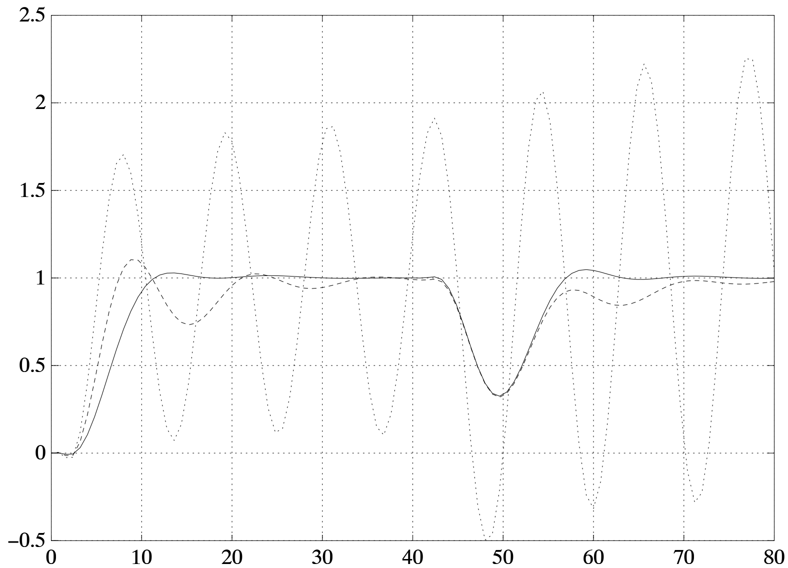



**Figure 7.7**

# 8. Lead-Lag Compensation

**8.1** Consider the following frequency domain specifications for a *closed* loop control system

**A** $|G(0)|$

**B** Bandwidth

**C** Resonance peak

together with the specifications in the singularity plot

**D** The distance between the dominant poles and the origin

**E** The number of poles

**F** The angle $\varphi$ which is made up by the position vector of the dominant poles and the negative real axis

For each of the groups above, which specifications are *foremost* associated with

**a.** The speed of the system.

**b.** The overshoot at reference steps.

**8.2** A second order system has the Bode plot shown in figure 8.1.

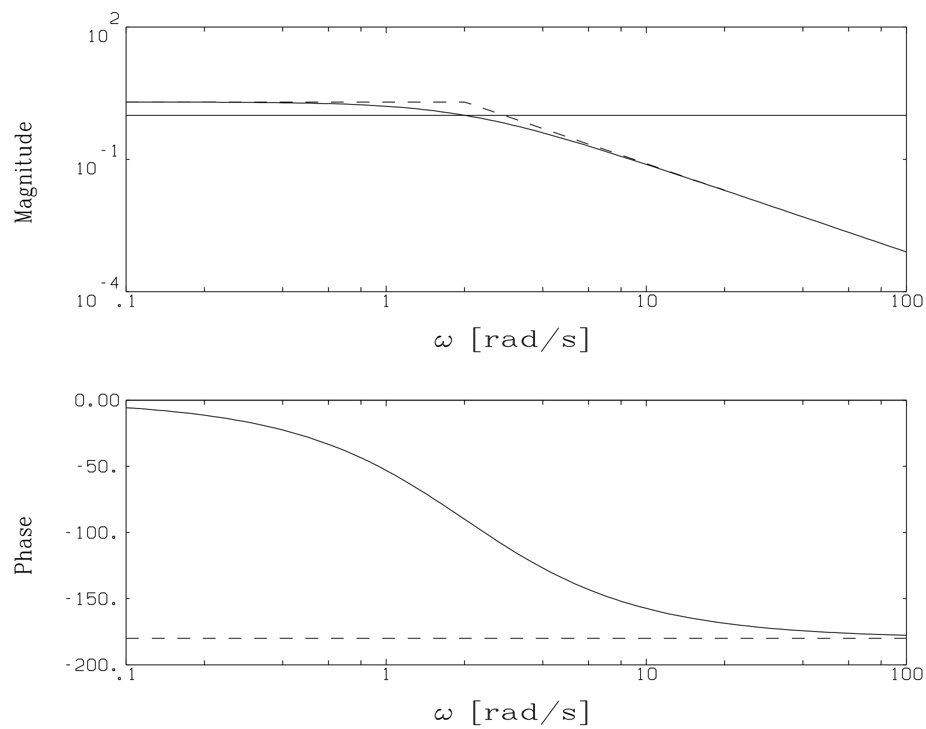



**Figure 8.1** Bode plot of the system in assignment 8.2.

We would like to connect a link $G_2$ in series with the system, in order to increase the speed of the closed loop system. The cross-over frequency, $\omega_c$, (the angle for which $|G| = 1$) is used as a measure of the system's speed. Which of the following $G_2$-candidates yield a faster system?

**A** $G_2 = K,\ K > 1$

**B** $G_2 = \dfrac{1}{s+1}$

**C** $G_2 = \dfrac{s+1}{s+2}$

**D** $G_2 = e^{-sT},\ T > 0$

**8.3** A system has the transfer function

$$G_P(s) = \frac{1}{s(s+1)(s+2)}$$

The system is part of a feedback loop together with a proportional controller with gain $K = 1$. The control error of the resulting closed loop system exhibits the following behavior: $e(t) \to 0,\ t \to \infty$ when the input is a step and $e(t) \to 2,\ t \to \infty$ when the input is a ramp.

Design a compensation link $G_k(s)$ which together with the proportional controller decreases the ramp error to a value less than 0.2. Also, the phase margin must not decrease by more than 6°.

**8.4** Consider a system with the following transfer function

$$G_P(s) = \frac{1.1}{s(s+1)}$$

A proportional controller with gain $K = 1$ is used to close the loop. However, the closed loop system becomes too slow. Design a compensation link, $G_k(s)$, that roughly doubles the speed of the closed loop system, without decreasing its phase margin.

**8.5** Consider the system

$$G_1(s) = \frac{1}{s(s+1)(s+2)}$$

If controlled by a proportional controller with gain $K = 1$, the stationary error of the closed loop system is $e = 0$ for a step input ($r = 1,\ t > 0$) and $e = 2$ for a ramp input ($r = t,\ t > 0$). One wants to increase the speed of the system by a factor 3, without compromising its phase margin or the ability to eliminate stationary errors. Device a compensation link $G_k(s)$ that fulfils the above criteria.

**8.6** A servo system has the open loop transfer function

$$G_o(s) = \frac{2.0}{s(s+0.5)(s+3)}$$

The system is subject to simple negative feedback and has a step response according to figure 8.2. As seen from the figure, the system is poorly damped and has a significant overshoot. The speed however, is satisfactory. The stationary error of the closed loop system with a ramp input is $e_1 = 0.75$.

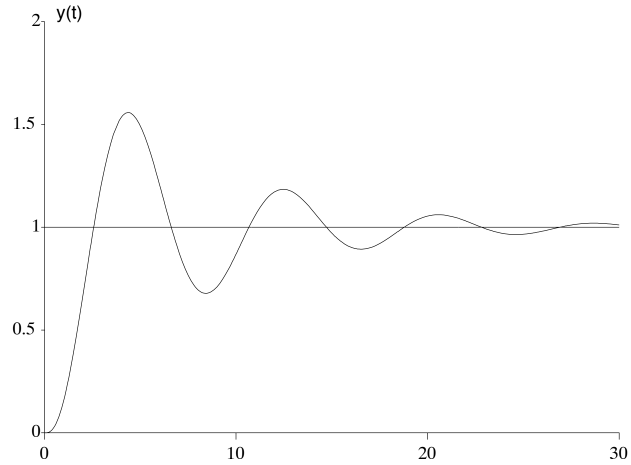



**Figure 8.2** Step response of the closed loop servo system in assignment 8.6.

Design a compensation link that increases the robustness of the system. Do this by increasing the phase margin to $\phi_m = 50°$ without affecting the speed of the system. ($\phi_m = 50°$ yields a relative damping $\zeta \approx 0.5$ which corresponds to an overshoot $M \approx 17\%$.) The stationary ramp error of the compensated system must not be greater than $e_1 = 1.5$.

**8.7** Consider a system with the open loop transfer function

$$G_1(s) = \frac{1.5}{s(s^2 + 2s + 2)}$$

The system is subject to simple negative feedback. The settling time (5%) is $T_s = 8.0$ *s*, the overshoot is $M_o = 27\%$, and the stationary ramp error $(r(t) = t)$ is $e_1 = 1.33$.

Device a phase lag compensation link

$$G_k(s) = K\frac{s + a}{s + a/M}$$

such that the stationary ramp error of the closed loop system is decreased to $e_1 = 0.1$, while speed and damping (stability) are virtually sustained.

# 9. State Feedback and Kalman Filtering

**9.1** A linear dynamical system with transfer function $G(s)$ is given. The system is controllable. Which of the following statements are unquestionably true?

**a.** The poles of the closed loop system's transfer function can be arbitrarily placed by means of feedback from all states.

**b.** The zeros of the closed loop system's transfer function can be arbitrarily placed by means of feedback from all states.

**c.** If the state variables are not available for measurements, they can always be estimated by diffrentiating the system output.

**d.** If the state vector is estimated by a Kalman filter

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

one can obtain an arbitrarily fast convergence of the estimate $\hat{x}$ towards the actual state vector $x$, by choice of the matrix $K$.

**9.2** Determine a control law $u = l_r r - Lx$ for the system

$$\frac{dx}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

such that the poles of the closed loop system are placed in $-4$ and the stationary gain is 1.

**9.3** The position of a hard drive head is described by the state space model

$$\frac{dx}{dt} = \begin{pmatrix} -0.5 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 3 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

**a.** Determine a state feedback

$$u = -Lx + l_r r$$

which places the poles of the closed loop system in $s = -4 \pm 4i$ and results in static gain 1 from reference to output.

**b.** Determine a Kalman filter

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

for the system. Briefly motivate necessary design choices.

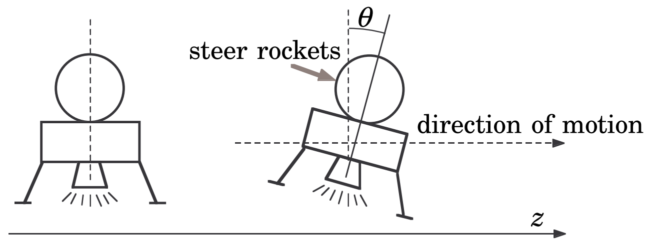



**Figure 9.1** The lunar lander in assignment 9.4.

**9.4** Figure 9.1 shows the lunar lander LEM of the Apollo project. We will study a possible system for controlling its horizontal movement above the moon surface.

Assume that the lander floats some distance above the moon surface by means of the rocket engine. If the angle of attack (the angle of the craft in relation to the normal of the moon surface) is nonzero, a horizontal force component appears, yielding an acceleration along the moon surface.

Study the block diagram in figure 9.2 showing the relation between the control signal $u$ of the rocket engine, the angle of attack, $\theta$, and the position $z$. The craft obeys Newton's law of motion in both the $\theta$

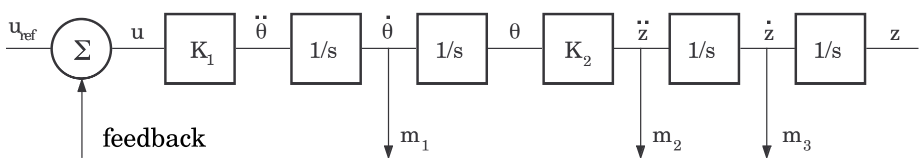



**Figure 9.2** Block diagram of the lander dynamics along the $z$-axis.

and $z$ directions. The transfer function from the astronaut's control signal $u_{ref}$ to the position $z$ is

$$G_z(s) = \frac{k_1 k_2}{s^4}$$

and it is practically impossible to manually maneuver the craft. To facilitate the astronaut's maneuvering task, we alter the craft dynamics by introducing an internal feedback loop. We are in possession the following measurement signals:

$m_1$ The time derivative of the attack angle, measured by a rate gyro.

$m_2$ The acceleration in the $z$ direction, measured by accelerometers mounted on a gyro-stabilized platform.

$m_3$ The speed in the $z$ direction, measured by Doppler radar.

Design a feedback controller which utilizes these measurements, and results in a closed loop system with three poles in $s = -0.5$, and lets the control signal of the astronaut shall be the *speed* reference in the $z$ direction.

**9.5** A conventional state feedback law does note guarantee integral action. The following procedure is a way of introducing integral action. Let the nominal system be

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

Augment the state vector with an extra component

$$x_{n+1} = \int^t e(s)\,ds = \int^t (r(s) - y(s))\,ds$$

The obtained system is described by

$$\frac{dx_e}{dt} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix} x_e + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

where

$$x_e = \begin{pmatrix} x \\ x_{n+1} \end{pmatrix}$$

A state feedback law for this system results in a control law of the form

$$u = -Lx - l_{n+1}x_{n+1} = -L_e x_e$$

This controller, which steers $y$ towards $r$, obviously has integral action. Use this methodology in order to determine a state feedback controller with integral action for the system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

such that the closed loop system obtains the characteristic polynomial

$$(s + \alpha)\left(s^2 + 2\zeta\omega s + \omega^2\right) = 0$$

**9.6** Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

One wishes to estimate the state variables by means of the model

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

Determine $K$ such that the poles of the Kalman filter are placed in $s = -4$.

**9.7** Consider the dynamical system

$$\frac{dx}{dt} = \begin{pmatrix} -4 & -3 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 3 \end{pmatrix} x$$

One desires a closed loop system with all poles in $-4$.

**a.** Assign feedback gains to all states such that the closed loop system obtains the desired feature.

**b.** Assume that only the output $y$ is available for measurement. In order to use state feedback, the state $x$ must be first be estimated by means of e.g. a Kalman filter, yielding the estimate $\hat{x}$. Subsequently, the control law $u = -L\hat{x}$ can be applied.

Is it possible to determine a Kalman filter for which the estimation error decreases according to the characteristic polynomial $(s+6)^2$?

**c.** Is it possible to determine a Kalman filter for which the estimation error decreases according to the characteristic polynomial $(s+3)^2$?

Briefly comment the obtained results.

**9.8** Consider the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = x_1$$

**a.** Design a state feedback $u = \ell_r r - \ell_1 x_1 - \ell_2 x_2$ which yields a closed loop system with static gain 1 and characteristic equation

$$s^2 + 2\zeta\omega s + \omega^2 = 0$$

**b.** Determine a Kalman filter

$$\dot{\hat{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} (y - \hat{x}_1)$$

where the dynamics of the estimation error have the characteristic equation $(s+a)^2 = s^2 + 2as + a^2 = 0$.

**c.** Give the equations of the controller which is obtained when the Kalman filter is combined with a state feedback.

**d.** Introduce the state variables $x$ and $\tilde{x}$ where $\tilde{x} = x - \hat{x}$ and write the closed loop system on state space form. Also give the characteristic equation of the closed loop system.

**e.** Write down the transfer function from $r$ to $y$.

**9.9** Consider the system

$$\dot{x} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

**a.** Design a state feedback $u = -\ell_1 x_1 - \ell_2 x_2$ which yields a closed loop system

$$s^2 + 2\sqrt{2}s + 4 = 0$$

Also state a Kalman filter

$$\dot{\hat{x}} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} (y - \hat{x}_1)$$

which has the characteristic equation $s^2 + 6s + 9 = 0$.

**b.** Give the output feedback (state space) equations which are obtained when the Kalman filter is combined with the state feedback.

**c.** Introduce the state variable $x$ and $\tilde{x} = x - \hat{x}$. Write down the characteristic equation of the closed loop system.

**d.** Consider the controller as a system with input $y$ and output $u$. Give the transfer function of this system.

**9.10** Consider a process consisting of a ball, rolling on a beam, according to figure 9.3.

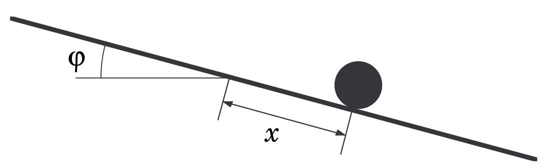



**Figure 9.3** The beam process in assignment 9.10.

The process can be described by the equation

$$\frac{d^2x}{dt^2} = k\varphi$$

where $x$ is the position of the ball and $\varphi$ is the angle of inclination of the beam. With $k = 1$ the following state space model is obtained

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

where $u$ is the inclination of the beam, $x_1$ is the position of the ball and $x_2$ its speed.

However, it is difficult to calibrate the system so that zero input yields zero output. In order to model this, we introduce an unknown zero input error in the control signal. The model becomes

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (u + a)$$

We can try to eliminate the zero input error $a$ by means of a Kalman filter in the following way: Introduce $a$ as an extra state variable $x_3$. Since $a$ is constant we have

$$\frac{dx_3}{dt} = 0$$

and the augmented system can be described by the following state space model

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

**a.** Is this method of auto-calibration is practically feasible? Why/why not?

**b.** Design a Kalman filter with the characteristic equation

$$(s + \alpha)(s^2 + 2\zeta\omega s + \omega^2) = 0$$

Give the equations of the Kalman filter and try to interpret their meaning intuitively.

# 10. Controller Structures

**10.1** Figure 10.1 shows a block diagram of the temperature control system in a house. The reference temperature (the thermostat set point) is given by $r$, the output $y$ is the indoor temperature and the disturbance $d$ is due to the outdoor temperature. The transfer function $G_1(s)$ represents the dynamics of the heating system and $G_2(s)$ represents the dynamics of the air inside the house. The controller $G_R$ is a P controller with gain $K = 1$.

Assume that the influence $d$ of the outdoor temperature can be exactly measured. Determine a feedforward link $H$, such that the indoor temperature becomes independent of the outdoor temperature. What is required in order to obtain a good result from the feedforward?

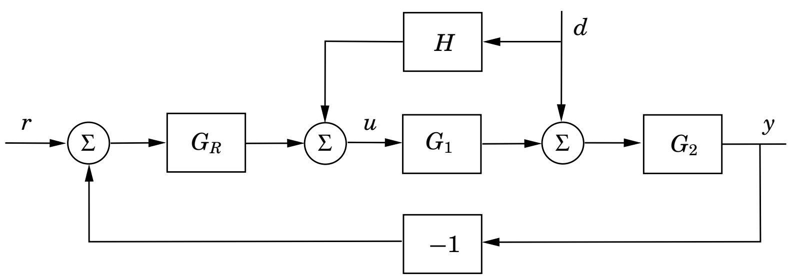



**Figure 10.1** Block diagram of the temperature control system in a house.

**10.2** Figure 10.2 shows a block diagram of a level control system for a tank. The inflow $x(t)$ of the tank is determined by the valve position

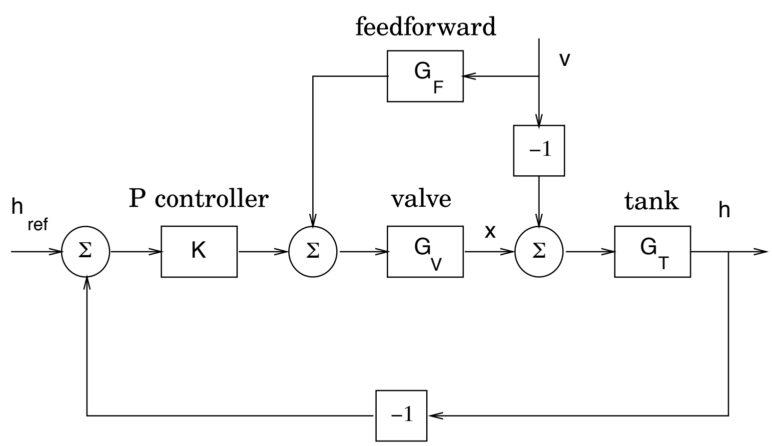



**Figure 10.2** Block diagram of the level control system in assignment 10.2.

and the outflow $v(t)$ is governed by a pump. The cross section of the tank is $A = 1\ m^2$.

The assignment is to control the system so that the level $h$ in the tank is held approximately constant despite variations in the flow $v$. The transfer function of the valve from position to flow is

$$G_v(s) = \frac{1}{1 + 0.5s}$$

The tank dynamics can be determined through a simple mass balance.

**a.** Assume that $G_F = 0$, i.e. that we don't have any feedforward. Design a P controller such that the closed loop system obtains the characteristic polynomial $(s + \omega)^2$.

How large does $\omega$ become? What stationary level error is obtained after a 0.1 step in $v(t)$?

**b.** Design a PI controller which eliminates the stationary control error otherwise caused by load disturbances.

Determine the controller parameters so that the closed loop system obtains the characteristic polynomial $(s + \omega)^3$. How large does $\omega$ become?

**c.** To further decrease the influence of load disturbances, we introduce feedforward based on measurements of $v(t)$. Design a feedforward controller $G_F$ that eliminates the influence of outflow variations by making corrections to $x(t)$.

*Comment.* As all variables describe deviations from the operation point, the reference value for the level $h$ can be set to zero.

**10.3** Consider the system in figure 10.3. The transfer function of the process is given by

$$G_P(s) = \frac{1}{s+3}$$

and $G_R(s)$ is a PI controller with transfer function

$$G_R(s) = K(1 + \frac{1}{ST_i})$$

$K_f$ is a constant feedforward from the reference signal $r$.

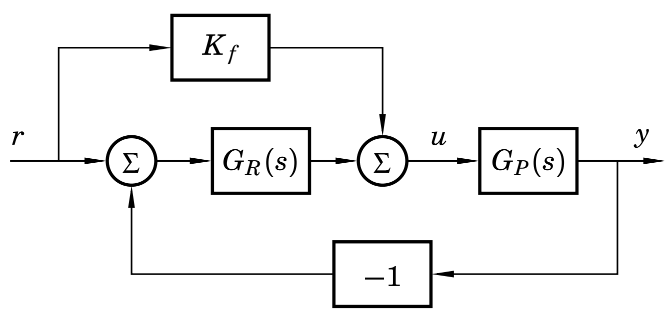



**Figure 10.3** Block diagram showing assignment 10.3.

**a.** Let $K_f = 0$ and determine $K$ and $T_i$ such that the poles of the closed loop system are placed in $-2 \pm 2i$, which is assessed to supress disturbances well.

**b.** Discuss the influence of the feedforward on the system's response to reference changes.

The closed loop transfer function of the system has one zero. Eliminate it by choosing an appropriate constant feedforward $K_f$.

**10.4** The system in assignment 10.3 can be described by an equivalent block diagram, according to figure 10.4. Write down the transfer

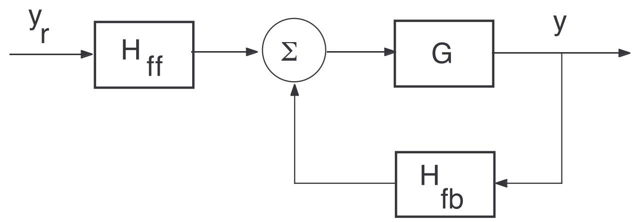



**Figure 10.4** Equivalent block diagram in assignment 10.3.

functions $H_{ff}(s)$ and $H_{fb}(s)$. Discuss the result and consider the effect of the feedforward when the controller contains a D term.

**10.5** The block diagram in figure 10.5 shows cascade control of a tank. The transfer function $G_1$ describes a valve whereas the transfer

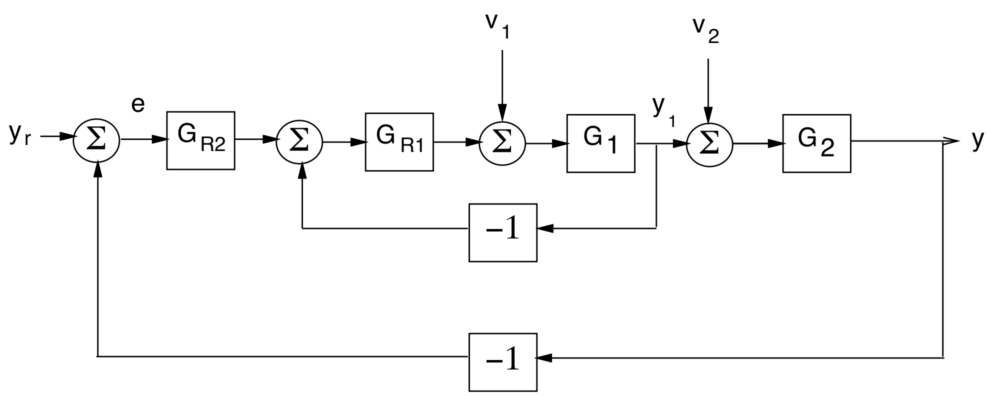



**Figure 10.5** The cascade in assignment 10.5.

function $G_2$ describes the dynamics of the tank. The objective is to control the tank level $y$. This is done by controlling the valve $G_1$ in an inner control loop, whereas $y$ is controlled by an outer control loop. Both the control loops are cascaded so that the reference of the inner loop is the output of the controller in the outer loop.

There are two disturbances in the system, namely the disturbance flow $v_2$, which is added to the controlled flow $y_1$ and pressure variations $v_1$ in the flow before the valve. Discuss the choice of controller (P or PI) in the inner and outer loop, respectively, with respect to elimination of stationary control errors.

**10.6** Consider figure 10.5 and assume that $G_1(s) = \frac{2}{s+2}$ describes a valve whereas $G_2(s) = \frac{1}{s}$ describes a tank.

**a.** Determine a P controller $G_{R1}(s) = K_1$ such that the inner control loop becomes 5 times faster than the uncontrolled valve.

**b.** Design a PI controller$G_{R2}(s) = K_2(1 + \frac{1}{T_i s})$ for the outer loop, which yields a system at least 10 times as slow as the closed inner loop. Approximate the inner loop by $G_{\text{inner}}(s) \approx G_{\text{inner}}(0)$.

**10.7** In a certain type of steam boiler, a dome is used to separate the steam from the water (see figure 10.6). It is essential to keep the dome level constant after load changes. The dome can be described

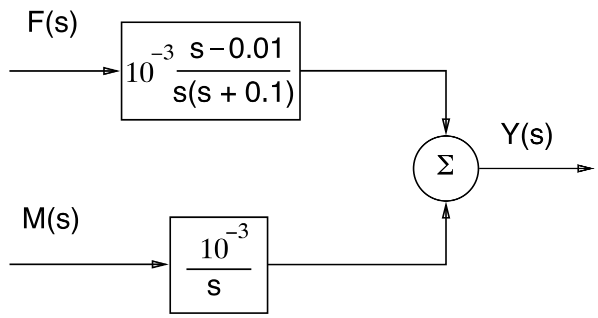



**Figure 10.6** Block diagram of steam boiler with dome.

by the model

$$Y(s) = \frac{10^{-3}}{s}M(s) + \frac{s-0.01}{s(s+0.1)}10^{-3}F(s)$$

where $Y$ is the dome level [m], $M$ is the feed water flow [kg/s] and $F$ is the steam flow [kg/s].

**a.** Assume a constant steam flow. Design a P controller, controlling the feed water flow by measuring the dome level. Choose the controller parameters such that the control error caused by a step in the dome level goes down to 10 % of its initial value after 10 seconds.

**b.** Consider the closed loop system. Write down the stationary level error $Y$ caused by a step disturbance of 1 kg/s in the steam flow $F$.

**c.** Consider the initial system. Determine a feedforward link $H(s)$ from steam flow $F(s)$ to feed water flow $M(s)$, such that the level $Y$ becomes independent of changes in the steam flow.

**10.8** Consider the system in figure 10.7. Assume that the disturbance $d$ is measurable and that it does not contain frequency components above 5 rad/s. Write down a feedforward link $H(s)$ which eliminates at least 90% of the disturbance $d$. The transfer functions are $G_1(s) = 1/(s+1)$ and $G_2(s) = 1/s$.

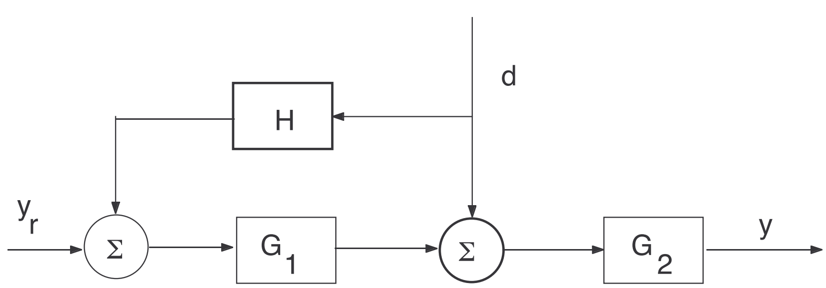



**Figure 10.7** Block diagram in assignment 10.8.

**10.9** Assume that a servo motor

$$G_P(s) = \frac{1}{s(s+1)}$$

is controlled by the P controller $G_R(s) = 2$. What is the delay margin of the system?

**10.10** Consider the same process and controller as in the previous assignment. Now the process is controlled over a very slow network which introduces a one second delay in the control loop. In order to deal with this problem an Smith predictor is utilized, see figure 10.8.

**a.** Assume that the model and the process are identical. What are the transfer functions for the blocks (*Controller*, *Process*, *Model*, *Model with no delay*) in our example?

**b.** The block diagram of the Smith predictor can be redrawn according to figure 10.9. What is the transfer function of the Smith predictor (from $e$ to $u$) in our example?

**c.** Use the approximation $e^x \approx 1 + x$ in order to simplify the transfer function of the controller. Compare the controller to compensation links.

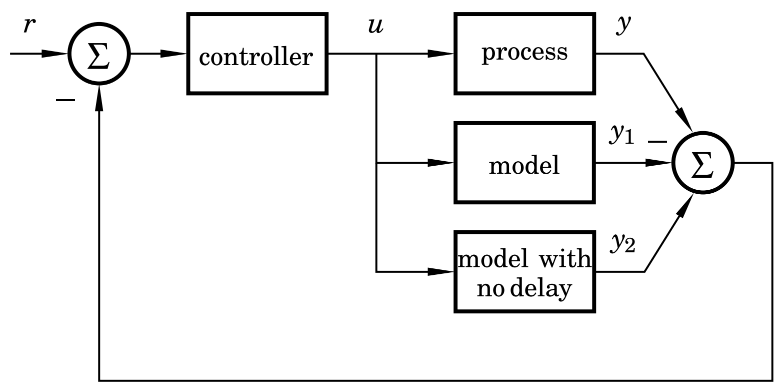



**Figure 10.8** Working principle of the Smith predictor.

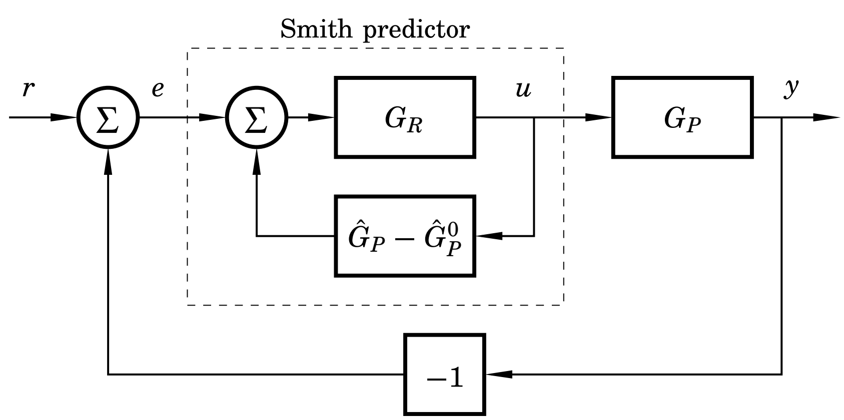



**Figure 10.9** Block diagram equivalent to figure 10.8.

**10.11** Figure 10.10 shows the result of a frequency analysis carried out on the beam (a part of the 'ball on the beam' process). One sees that the process dynamics can be well approximated by an integrator, for low frequencies. One also sees that for high frequencies, the phase curve diverges in a way which resembles a delay. Consequently, it would be possible to describe the process by

$$G(s) = \frac{k}{s} e^{-sL}$$

Use the Bode plot in order to determine approximate values of the gain $k$ and delay $L$.

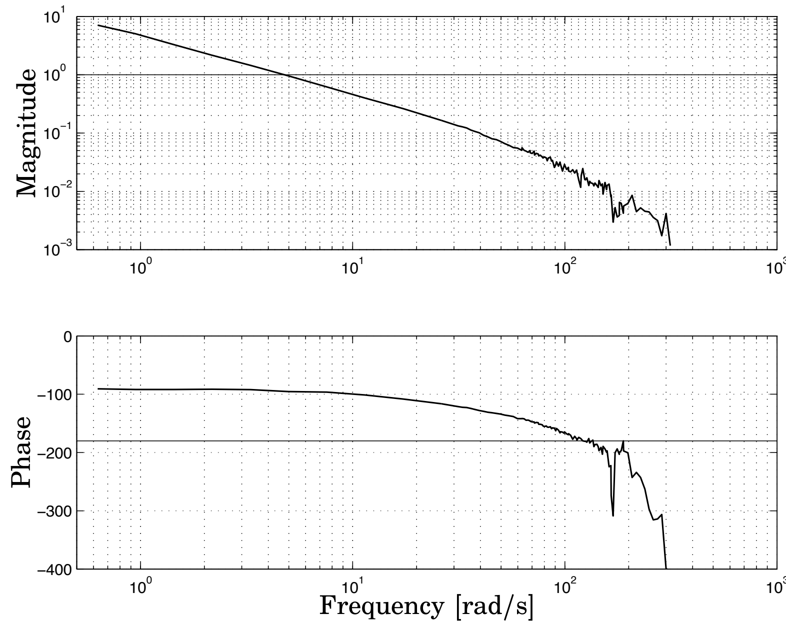



**Figure 10.10** Measured Bode plot of the beam.

# 11. Design Examples

## 11.1 Depth Control of Submarine

### Purpose

This assignment deals with depth control of a submarine from the forties. Two control methods are tested – PD and state feedback. The latter method was used in reality.

### Background

Depth control of submarines can be achieved by means of varying the rudder angle $\beta$ according to figure 11.1. The depth $h$ is measured by

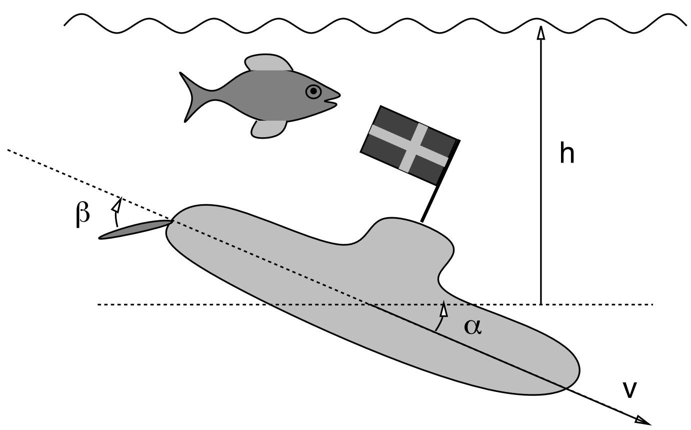



**Figure 11.1** Depth control of the submarine in assignment 11.1.

means of a manometer. By manually generating a sinusoidal rudder angle $\beta$ (by means of a table and watch — don't forget that this was the end of the forties) one can use frequency analysis to estimate the transfer function $G(s)$ from $\beta$ to $h$ (for a constant speed $v$). The resulting Bode plots for three different speeds are shown in figure 11.2.

### Specifications

In this case no specifications were given except "Make it as good as possible".

### Problem Formulation

Assume that the speed is $v = 3$ knots. The problem lies in computing a control law which gives a satisfactory settling of the depth $h$ for the given speed. This does not guarantee equally satisfactory results at other speeds.
In an initial approach one wanted to control the depth $h$ of the submarine, solely based on measurements of $h$.

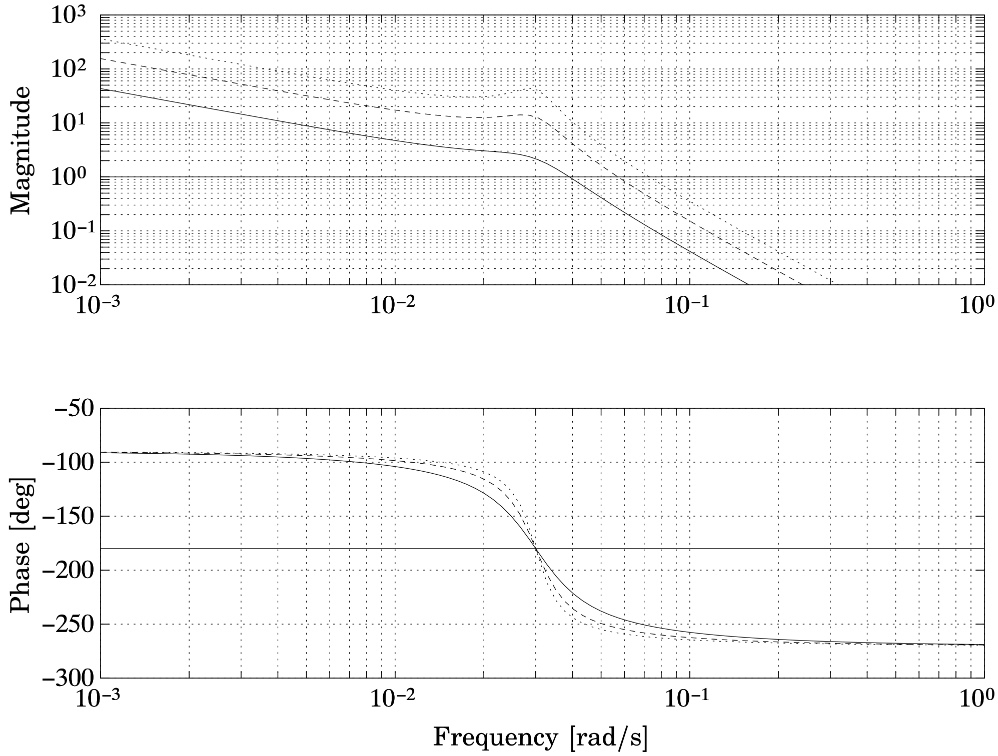



**Figure 11.2** Bode plot of the estimated transfer function $G(s)$ from $\beta$ [deg] to $h$ [m] in assignment 11.1 for the speeds $v = 3$ (solid curves), 5 (dashed curves) and 7 knots (dotted curves).

**a.** What is the maximal allowed gain $K$ in order to achieve a stable closed loop system with a P controller $\beta = K(h_{\text{ref}} - h)$. Use the Bode plot in figure 11.2?

**b.** One wants to obtain a cross-over frequency $\omega_c = 0.03$ rad/s, using a PD controller $G_r(s) = K(1 + T_D s)$. How shall $K$ and $T_D$ be chosen in order to obtain a 60° phase margin $\phi_m$?

**c.** How is the stability of the closed loop system in (b) affected if the speed is increased from 3 to 7 knots? Suggest different ways in which speed variations can be taken into consideration.

For angular frequencies above 0.05 rad/s one can use the approximation

$$\begin{cases} G_{\alpha\beta}(s) = \dfrac{k_v}{s^2} \\ G_{h\alpha}(s) = \dfrac{v}{s} \end{cases} \tag{11.1}$$

where $G_{\alpha\beta}(s)$ and $G_{h\alpha}(s)$ are the transfer functions from $\beta$ to $\alpha$ and from $\alpha$ to $h$, respectively (see figure 11.3). The constant $k_v$ depends on the speed $v$.

**d.** Determine $k_v$ by means of the Bode plot in figure 11.2. (1 knot $\approx$ 1.852 km/h $= 1.852/3.6 \approx 0.514$ m/s.)

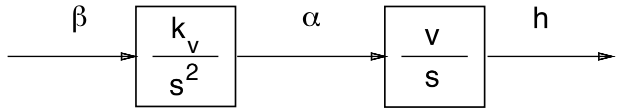



**Figure 11.3** Block diagram of a submarine model which is valid for $\omega >$ 0.05 rad/s.

**e.** Assume that the approximate model

$$G_{h\beta}(s) = \frac{k_v v}{s^3}$$

is under P control $\beta = K(h_{ref} - h)$. Determine which values of $K$ that yield an asymptotically stable system. Does this concur with the results obtained in sub-assignment a?

One can improve the performance of the control system by utilizing additional feedback form the trim angle $\alpha$ and its derivative $d\alpha/dt$.

**f.** Introduce the states $x_1 = d\alpha/dt$, $x_2 = \alpha$ and $x_3 = h$ together with the input $u = \beta$. Use the control law $u = u_r - l_1 x_1 - l_2 x_2 - l_3 x_3 = u_r - Lx$ and determine $L$ such that the characteristic equation of the closed loop system becomes

$$(s + \gamma\omega_0)(s^2 + 2\zeta\omega_0 s + \omega_0^2) = 0$$

**g.** The reference $h_{\text{ref}}$ for the depth $h$ is introduced according to

$$u_r = L_r h_{\text{ref}}$$

How shall $L_r$ be chosen in order to obtain $h = h_{\text{ref}}$ in stationarity?

One decided to choose $\zeta = 0.5$ and $\gamma = 2$ which was considered to give an adequately damped step response. However, the choice of $\omega_0$ requires some further thought. It should not be chosen too low, since the approximate model (11.1) is only valid for $\omega > 0.05$ rad/s. On the other hand, choosing $\omega_0$ too high would result in large rudder angles caused by the large values of the coefficients $l_j, j = 1, 2, 3$.

**h.** How large can $\omega_0$ be chosen if a step disturbance in the manometer signal corresponding to $\Delta h = 0.1m$ should not give rise to larger rudder angles than 5°?

In the actual case $\omega_0 = 0.1$ rad/s was chosen. A semi-automatic system was evaluated first. The signal $u - u_r - l_1 x_1 - l_2 x_2 - l_3 x_3$ was displayed to an operator, who manually tried to keep the signal zero by means of the ordinary rudder servo. The control action was very satisfactory. Settling times of 30-60 s were obtained throughout the speed range. The complete automatic system was then evaluated on the Swedish submarine 'Sjöborren' (The Sea Urchin). The accuracy during march in calm weather was ±0.05 m.

## 11.2 Control of Elastic Servo

### Purpose

The aim of the assignment is to control the angular speed of a flywheel which is connected to another flywheel by a weak axis. The second flywheel is driven by a motor. Different control strategies are evaluated and compared with respect to performance.

### Background

Figure 11.4 shows a simplified model of an elastic servo. It could also constitute a model of a weak robot arm or an elastic antenna system mounted on a satellite. The turn angles of the flywheels

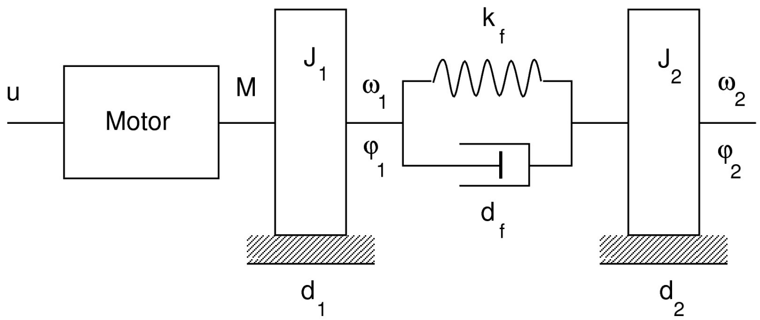



**Figure 11.4** Model of the elastic servo in assignment 11.2.

are denoted $\varphi_1$ and $\varphi_2$, respectively, whereas $\omega_1 = \dot{\varphi}_1$ and $\omega_2 = \dot{\varphi}_2$ denote the corresponding angular speeds. The flywheels have moments of inertia $J_1$ and $J_2$, respectively. They are connected by an axle with spring constant $k_f$ and damping constant $d_f$. The system is subject to bearing friction, which is represented by the damping constants $d_1$ and $d_2$. One of the flywheels is driven by a DC motor, which is itself driven by a current-feedback amplifier. The motor and amplifier dynamics are neglected. The momentum of the motor is proportional to the input voltage $u$ of the amplifier, according to

$$M = k_m \cdot I = k_m k_i u$$

where $I$ is the current through the rotor coils. Momentum equilibrium about the flywheel yields the following equations

$$\begin{cases} J_1\dot{\omega}_1 = -k_f(\varphi_1 - \varphi_2) - d_1\omega_1 - d_f(\omega_1 - \omega_2) + k_m k_i u \\ J_2\dot{\omega}_2 = +k_f(\varphi_1 - \varphi_2) - d_2\omega_2 + d_f(\omega_1 - \omega_2) \end{cases}$$

We introduce the state variables

$$\begin{cases} x_1 = \omega_1 \\ x_2 = \omega_2 \\ x_3 = \varphi_1 - \varphi_2 \end{cases}$$

and consider the angular speed $\omega_2$ as the output, i.e.

$$y = k_{\omega_2} \cdot \omega_2$$

This gives us the following state space model of the servo.

$$\dot{x} = Ax + Bu = \begin{pmatrix} -\frac{d_1+d_f}{J_1} & \frac{d_f}{J_1} & -\frac{k_f}{J_1} \\ \frac{d_f}{J_2} & -\frac{d_f+d_2}{J_2} & \frac{k_f}{J_2} \\ 1 & -1 & 0 \end{pmatrix} x + \begin{pmatrix} \frac{k_m k_i}{J_1} \\ 0 \\ 0 \end{pmatrix} u$$

$$y = Cx = \begin{pmatrix} 0 & k_{\omega_2} & 0 \end{pmatrix} x$$

The following values of constants and coefficients have been measured and estimated for a real lab process.

$$\begin{aligned}
J_1 &= 22 \cdot 10^{-6}\,\text{kgm}^2 \\
J_2 &= 65 \cdot 10^{-6}\,\text{kgm}^2 \\
k_f &= 11.7 \cdot 10^{-3}\,\text{Nm/rad} \\
d_f &= 2e-5 \\
d_1 &= 1 \cdot 10^{-5}\,\text{Nm/rad/s} \\
d_2 &= 1 \cdot 10^{-5}\,\text{Nm/rad/s} \\
k_m &= 0.1\,\text{Nm/A} \\
k_i &= 0.027\,\text{A/V} \\
k_{\omega_1} &= k_{\omega_2} = 0.0167\,\text{V/rad/s}
\end{aligned}$$

### Problem Formulation

The input is the voltage $u$ over the motor and we want to control the angular speed $\omega_2$ of the outer flywheel.
It is desired to quickly be able to change $\omega_c$, while limiting the control system's sensitivity against load disturbances and measurement noise. The system also requires active damping, in order to avoid an excessively oscillative settling phase.

### Specifications

**1.** The step response of the closed loop system should be fairly well damped and have a rise time of 0.1-0.3 s. The settling time to $\pm 2\%$ shall be at most 0.5 s. A graphical specification of the step response is given in figure 11.5.

**2.** Load disturbances must not give rise to any static errors.

**3.** Noise sensitivity should not be excessive.

### Ziegler-Nichols Method

The Bode plot of the transfer function from $u$ to $\omega_2$ is shown in figure 11.6.

**a.** Use Ziegler-Nichols frequency method in order to determine suitable PID parameters.
Ziegler-Nichols method often gives a rather oscillative closed loop system. However, the obtained parameters are often a reasonable starting point for manual tuning.

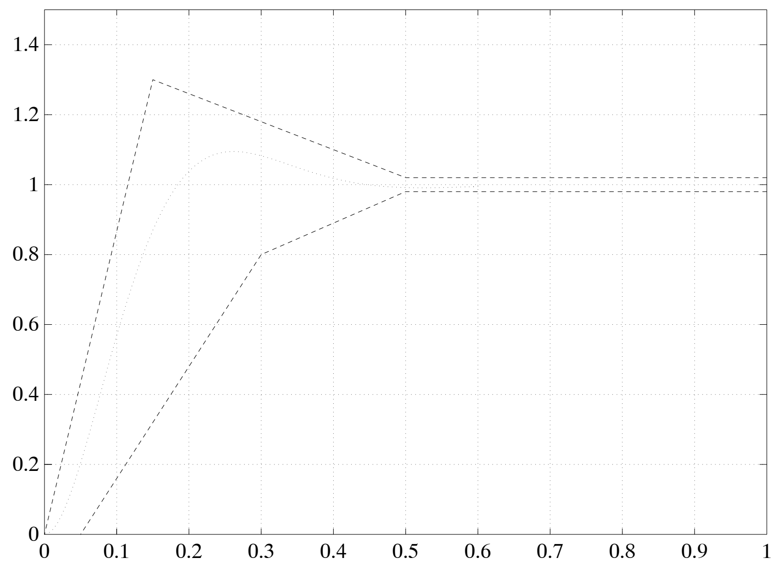



**Figure 11.5** The step response of the closed loop system shall lie between the dashed lines.

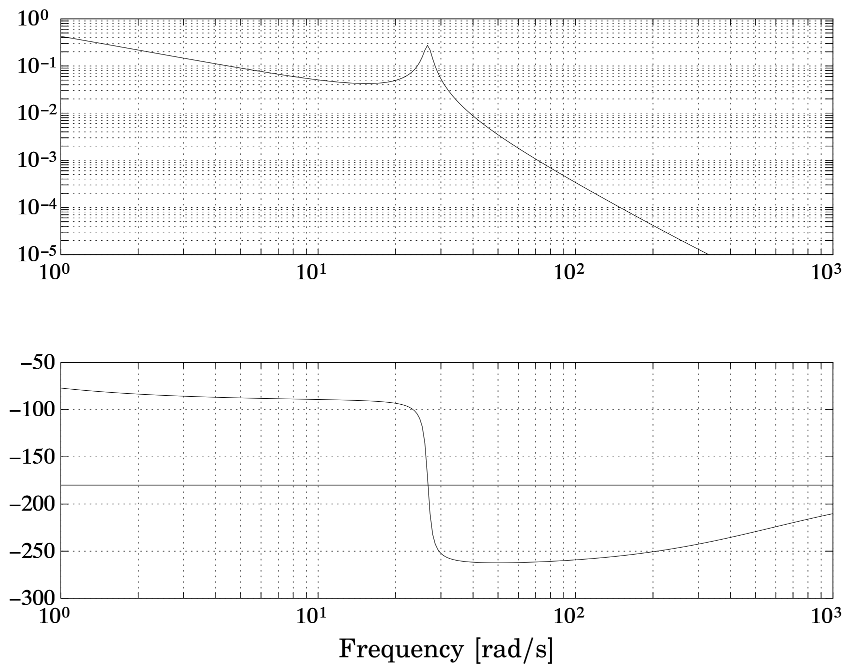



**Figure 11.6** Bode plot of the servo process.

## State Feedback and Kalman Filtering

If it is possible to measure all states, the poles of the closed loop system can be arbitrarily placed through the feedback control law

$$u(t) = -Lx(t) + L_r y_r(t)$$

if the system is also controllable.

**b.** The gain $L_r$ is chosen such that the stationary gain of the closed loop system becomes 1, i.e. $y = y_r$ in stationarity. The coefficient $L_r$ can hence be expressed in $L$ and $k_{\omega_2}$. Do this.
In order to meet specification 2, one must introduce integral action in the controller. One way to achieve this it thorough the control law

$$u(t) = -Lx(t) + L_r y_r(t) - L_i \int_{-\infty}^{t} (y(s) - y_r(s))ds$$

This can be interpreted as feedback from an 'extra' state $x_i$ according to

$$\begin{cases} \dot{x}_i = y - y_r \\ u = -Lx + L_r y_r - L_i x_i \end{cases}$$

Figure 11.7 shows a block diagram of the entire system

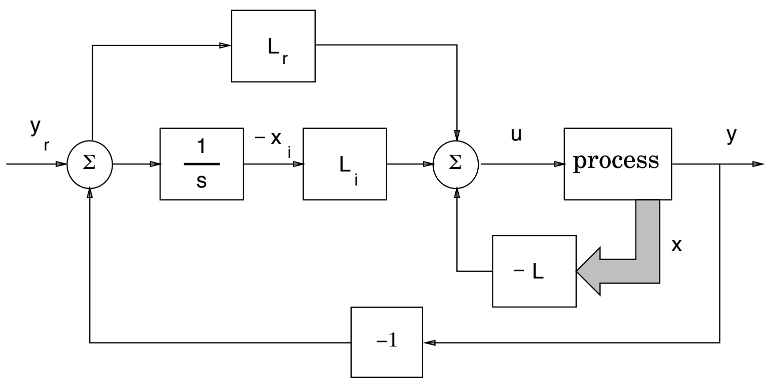



**Figure 11.7** Block diagram of the state feedback control system in assignment 11.2.

**c.** How does the augmented state space model look like? Introduce the notion $x_e$ for the augmented state vector.
Since the states are not directly measurable, they must be reconstructed in some way. A usual way is to introduce a Kalman filter

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

and then close the loop from the estimated states $\hat{x}$

$$u = -L\hat{x} + L_r y_r - L_i x_i$$

It is, however, unnecessary to estimate $x_i$ since we have direct access to this state. The block diagram of the entire system is shown in figure 11.8. Let $L'$ denote the augmented row matrix $\begin{pmatrix} L & L_i \end{pmatrix}$ and call the augmented system matrices $A'$ and $B'$, respectively. The problem consists in finding suitable $L$, $L_i$ and $K$ by placing the eigenvalues of $A' - B'L'$ and $A - KC$. Since the both eigenvalue problems are of a bit too high dimension for enjoyable hand calculations, we use Matlab to investigate a few choices of pole placements.

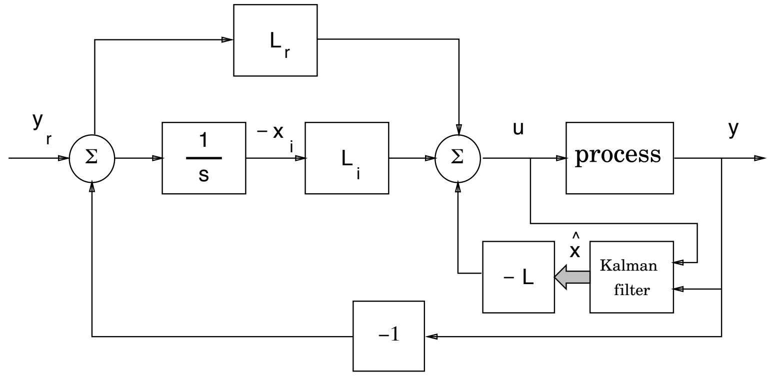



**Figure 11.8** Block diagram showing the Kalman filter and state feedback in assignment 11.2.

In order not to end up with too many free parameters, we place the poles in a Butterworth pattern. I.e. the poles are equally distributed on a half circle in the left half plane. We place the eigenvalues of $A' - B'L'$ on a half circle with radius $\omega_m$, whereas the eigenvalues of $A - KC$ are placed on a half circle with radius $\omega_o$ (see figure 11.9). A suitable $\omega_m$ can be obtained from Specification 1, i.e. that the

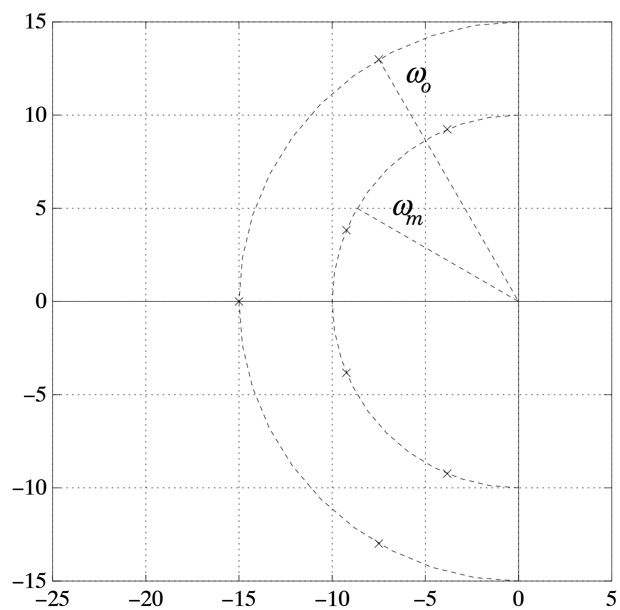



**Figure 11.9** The pole placement in assignment 11.2.

settling time $T_s$ to reach within 2% of the stationary value must be less than 0.5 s. A coarse estimation of $T_s$ for a second order system with relative damping $\zeta$ and natural frequency $\omega$ is given by

$$T_s \approx -\frac{\ln \epsilon}{\zeta \omega}$$

where $\epsilon$ is the maximal deviation from the final value. Since we have a 4$^{\text{th}}$ degree system, we cannot use this approximation directly.

If we, however, only consider the least damped pole pair ($\zeta = 0.38$ and $\omega = \omega_m$) in figure 11.9 we obtain

$$\omega_m \approx -\frac{\ln \epsilon}{T_s \zeta} \tag{11.2}$$

**d.** Which value of $\omega_m$ is obtained from the formula (11.2)?
We let $\omega_m = 20$ which yields $T_s < 0.5$ s. We can let $L_r = 0$ since we have integral action in the controller and thus stationary closed loop gain 1. Figure 11.10 shows the step response of the closed loop system for $L_r = 0$ and $L_r$ chosen according to sub-assignment b, respectively. By letting $L_r = 0$ the step response overshoot is suffi-

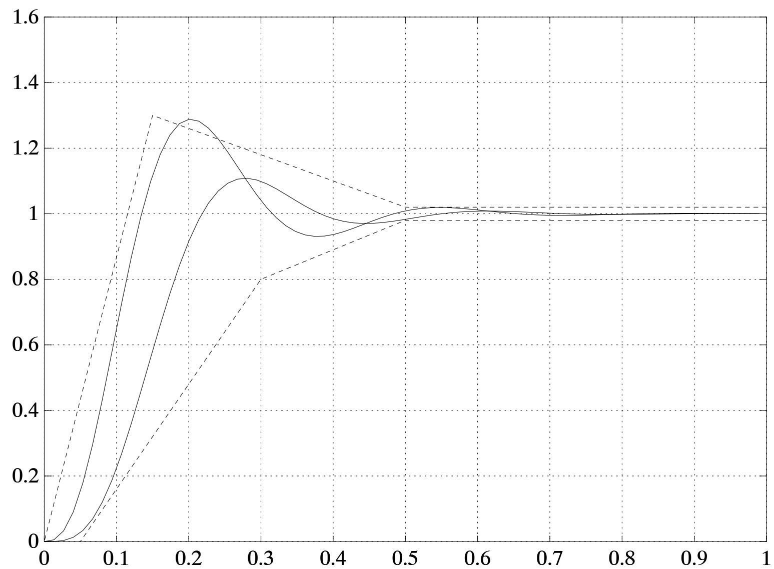



**Figure 11.10** Step response of the closed loop system for $\omega_m = 20$ rad/s. Choosing $L_r$ according to sub-assignment b yields the system with the larger overshoot. The other curve is the step response corresponding to $L_r = 0$.

ciently decreased to fulfill the specification.
We now fix $\omega_m$ and vary $\omega_o$. The following test shall be used to evaluate the control performance. At time $t = 0$ there is a unit step in the reference value $y_r$ followed by a load disturbance $d = -1$ in the control signal at $t = 1$ and the introduction of measurement noise (in $y$) at $t = 3$. The variance of the noise is 0.01. The result is shown in figure 11.11.

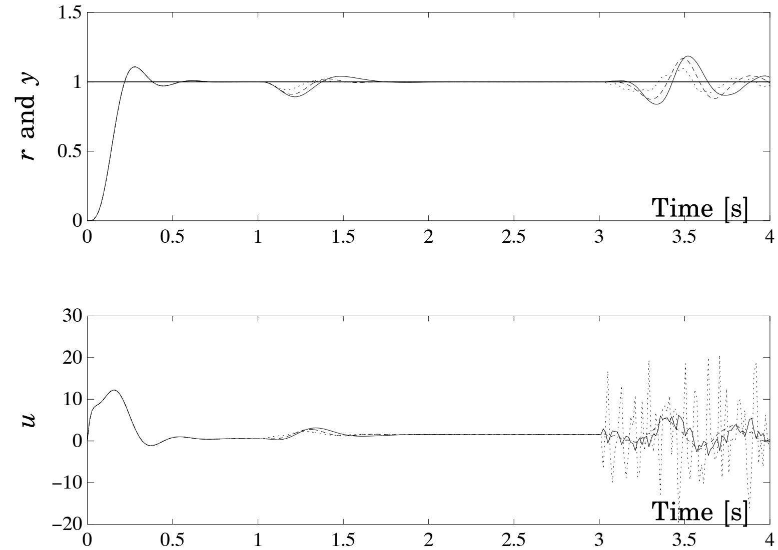



**Figure 11.11** Evaluation of control with $\omega_m = 20$ and $\omega_o = 10$ (solid curves), 20 (dashed curves) and 40 (dotted curves).

**e.** Which value of $\omega_o$ seems to be best when it comes to elimination of load disturbances? Which $\omega_o$ is best when it comes to suppressing measurement noise?

# Solutions to Chapter 1. Model Building and Linearization

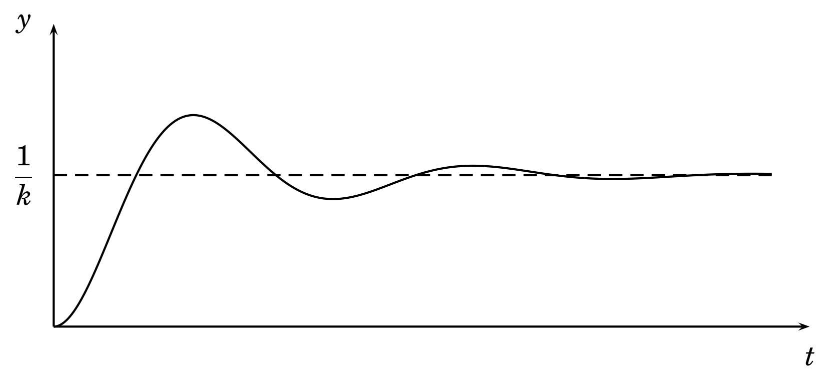
**1.1 a.** With $x_1 = y$ and $x_2 = \dot{y}$ the system is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{1}{m} \end{pmatrix} f$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

**b.** which has the solution

$$y(t) = \frac{1}{k}\left(1 - \frac{\sigma}{\omega_d} e^{-\sigma t} \sin \omega_d t - e^{-\sigma t} \cos \omega_d t\right)$$

where $\sigma = c/2m$ and $\omega_d = \sqrt{k/m - c^2/4m^2}$.




**1.2** With states $x_1 = v_{\text{out}}$ and $x_2 = \dot{v}_{\text{out}}$, the system is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{LC} \end{pmatrix} v_{\text{in}}$$
$$v_{\text{out}} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

**1.3 a.** We can choose e.g. the height $h$ as state variable. The volume change in the tank is given by

$$A\dot{h} = q_{\text{in}} - q_{\text{out}}$$

and from Torricelli's law we obtain $q_{\text{out}} = a\sqrt{2gh}$. The sought differential equation becomes

$$\dot{h} + \frac{a}{A}\sqrt{2gh} = \frac{1}{A} q_{\text{in}}$$

**b.**

$$\dot{h} = -\frac{a}{A}\sqrt{2gh} + \frac{1}{A}q_{\text{in}} \qquad (= f(h, q_{\text{in}}))$$
$$q_{\text{out}} = a\sqrt{2gh} \qquad (= g(h, q_{\text{in}}))$$

**c.** The outlflow must equal the inflow $q^0_{\text{out}} = q^0_{\text{in}}$. The level is calculated by letting $\dot{h} = 0$, which yields

$$h^0 = \frac{1}{2g}\left(\frac{q^0_{\text{in}}}{a}\right)^2$$

We determine the partial derivatives

$$\frac{\partial f}{\partial h} = -\frac{a}{A}\sqrt{\frac{g}{2h}} \qquad \frac{\partial f}{\partial q_{\text{in}}} = \frac{1}{A}$$
$$\frac{\partial g}{\partial h} = a\sqrt{\frac{g}{2h}} \qquad \frac{\partial g}{\partial q_{\text{in}}} = 0$$

By inserting $h = h^0$ above and introducing variables which denote deviations from the operating point: $\Delta h = h - h^0$, $\Delta q_{\text{in}} = q_{\text{in}} - q^0_{\text{in}}$, $\Delta q_{\text{out}} = q_{\text{out}} - q^0_{\text{out}}$ the linearized system is

$$\dot{\Delta h} = -\frac{a}{A}\sqrt{\frac{g}{2h^0}}\Delta h + \frac{1}{A}\Delta q_{\text{in}}$$
$$\Delta q_{\text{out}} = a\sqrt{\frac{g}{2h^0}}\Delta h$$

**d.** With $q_{\text{in}} = 0$ the nonlinear state space equation becomes

$$\dot{h} = -\frac{a}{A}\sqrt{2gh}, \quad h(0) = h^0$$

This is a separable differential equation with solution

$$h(t) = h^0\left(1 - \frac{a}{A}\sqrt{\frac{g}{2h^0}}t\right)^2, \quad 0 \le t \le \frac{A}{a}\sqrt{\frac{2h^0}{g}}$$

Letting

$$T = \frac{A}{a}\sqrt{\frac{2h^0}{g}},$$

enables us to write the solution on the more compact form

$$h(t) = h^0\left(1 - \frac{t}{T}\right)^2, \quad 0 \le t \le T$$

The outflow hence becomes

$$q_{\text{out}}(t) = a\sqrt{2gh^0}\left(1 - \frac{t}{T}\right) = q^0_{\text{out}}\left(1 - \frac{t}{T}\right), \quad 0 \le t \le T$$

In the linear state space equation we let $\Delta q_{\text{in}} = -q^0_{\text{in}}$ and obtain

$$\Delta\dot{h} = -\frac{1}{T}\Delta h - \frac{1}{A}q^0_{\text{in}}, \quad \Delta h(0) = 0$$

with solution

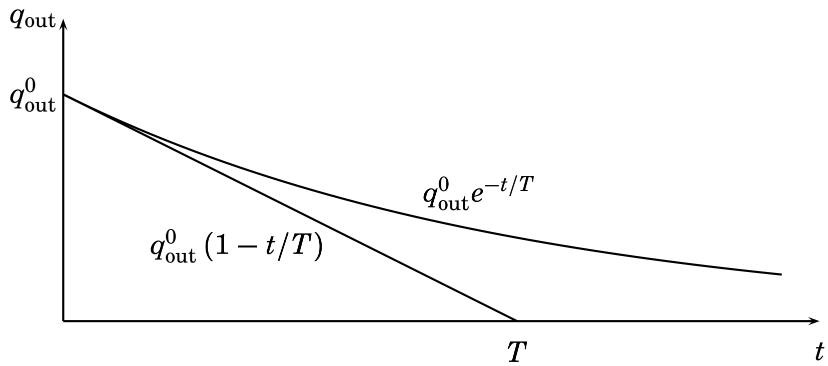
$$\begin{aligned}\Delta h(t) &= \frac{T}{A}q^0_{\text{in}}(e^{-t/T} - 1)\\ &= 2h^0(e^{-t/T} - 1), \quad t \geq 0\end{aligned}$$

and since

$$\Delta q_{\text{out}} = \frac{A}{T}\Delta h,$$

the outflow becomes

$$\begin{aligned}q_{\text{out}}(t) &= q^0_{\text{in}}(e^{-t/T} - 1) + q^0_{\text{out}}\\ &= q^0_{\text{out}}e^{-t/T}, \quad t \geq 0\end{aligned}$$




*Comment.* Observe that $T$ has the direct physical interpretations of the time it takes to empty the tank, but also shows up as scaling parameter in the solution of both the nonlinear and linear cases. Consequently, $T$ is a measure of the system's speed.

**1.4**

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}u$$

$$y = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

**1.5 a.**

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\sqrt{x_1} - x_1x_2 + u^2\\ y &= x_1\end{aligned}$$

**b.** A stationary point implies $\dot{x}_1 = \dot{x}_2 = 0$. From the first equation we directly obtain $x_2 = 0$. Subsequently, the second equation yields $\sqrt{x_1} = u^2$. Hence there are infinitely many stationary points and they can be parametrized through $t$ as $(x_1^0, x_2^0, u^0) = (t^4, 0, t)$.

**c.** $u^0 = 1$ gives the stationary point $(x_1^0, x_2^0, u^0) = (1, 0, 1)$. We let

$$\begin{aligned} f_1(x_1, x_2, u) &= x_2 \\ f_2(x_1, x_2, u) &= -\sqrt{x_1} - x_1 x_2 + u^2 \\ g(x_1, x_2, u) &= x_1 \end{aligned}$$

and compute the partial derivatives

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} = 0 && \frac{\partial f_1}{\partial x_2} = 1 && \frac{\partial f_1}{\partial u} = 0 \\ &\frac{\partial f_2}{\partial x_1} = -\frac{1}{2\sqrt{x_1}} - x_2 && \frac{\partial f_2}{\partial x_2} = -x_1 && \frac{\partial f_2}{\partial u} = 2u \\ &\frac{\partial g}{\partial x_1} = 1 && \frac{\partial g}{\partial x_2} = 0 && \frac{\partial g}{\partial u} = 0 \end{aligned}$$

At the stationary point we have

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} = 0 && \frac{\partial f_1}{\partial x_2} = 1 && \frac{\partial f_1}{\partial u} = 0 \\ &\frac{\partial f_2}{\partial x_1} = -\frac{1}{2} && \frac{\partial f_2}{\partial x_2} = -1 && \frac{\partial f_2}{\partial u} = 2 \\ &\frac{\partial g}{\partial x_1} = 1 && \frac{\partial g}{\partial x_2} = 0 && \frac{\partial g}{\partial u} = 0 \end{aligned}$$

After a variable substitution, the linearized system can be written

$$\begin{aligned} \begin{pmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Delta u \\ \Delta y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \end{aligned}$$

**1.6** At the sought operating point it holds that

$$\begin{aligned} 0 &= x_1^2 x_2 + 1 \\ 0 &= x_1 x_2^2 + 1 \\ y &= \arctan \frac{x_2}{x_1} + \frac{\pi^2}{8} \end{aligned}$$

which yields $x_1^0 = -1$, $x_2^0 = -1$ and $y^0 = \frac{\pi}{4} + \frac{\pi^2}{8}$. Computation of the partial derivatives now yields

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} = 2x_1 x_2 && \frac{\partial f_1}{\partial x_2} = x_1^2 && \frac{\partial f_1}{\partial u} = \sqrt{2}\cos u \\ &\frac{\partial f_2}{\partial x_1} = x_2^2 && \frac{\partial f_2}{\partial x_2} = 2x_1 x_2 && \frac{\partial f_2}{\partial u} = -\sqrt{2}\sin u \\ &\frac{\partial g}{\partial x_1} = \frac{-x_2}{x_1^2 + x_2^2} && \frac{\partial g}{\partial x_2} = \frac{x_1}{x_1^2 + x_2^2} && \frac{\partial g}{\partial u} = 4u \end{aligned}$$

With the variable substitution

$$
\begin{aligned}
\Delta u &= u - \frac{\pi}{4} \\
\Delta x_1 &= x_1 + 1 \\
\Delta x_2 &= x_2 + 1 \\
\Delta y &= y - \frac{\pi}{4} - \frac{\pi^2}{8}.
\end{aligned}
$$

the linearized system becomes

$$
\begin{aligned}
\begin{pmatrix} \dot{\Delta x_1} \\ \dot{\Delta x_2} \end{pmatrix} &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Delta u \\
\Delta y &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \pi \Delta u.
\end{aligned}
$$

**1.7 a.** Let $x_1 = y$ and $x_2 = \dot{y}$. The state space form becomes

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(1 + x_1^4)x_2 + \sqrt{u+1} - 2 \\
y &= x_1
\end{aligned}
$$

**b.**

$$
\begin{aligned}
\Delta \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \Delta x + \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} \Delta u \\
\Delta y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \Delta x
\end{aligned}
$$

where $\Delta x = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$ and $\Delta u = u - 3$, $\Delta x_1 = x_1 - 1$, $\Delta x_2 = x_2 - 0$ and $\Delta y = y - 1$.

**1.8 a.** The nonlinear state space equations are

$$
\begin{aligned}
\dot{x}_1 &= x_2 && = f_1(x_1, x_2, u) \\
\dot{x}_2 &= \omega^2 x_1 - \frac{\beta}{x_1^2} + u && = f_2(x_1, x_2, u) \\
y &= x_1 && = g(x_1, x_2, u)
\end{aligned}
$$

**b.** At stationarity it holds that

$$
\ddot{r}(t) = \omega^2 r_0 - \frac{\beta}{r_0^2} + 0 = 0
$$

i.e. $r_0^3 = \beta/\omega^2$. We now compute the partial derivatives

$$
\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial u} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \omega^2 + 2\beta/r_0^3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 3\omega^2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
$$

The linear system hence becomes

$$\frac{d\Delta x}{dt} = \begin{pmatrix} 0 & 1 \\ 3\omega^2 & 0 \end{pmatrix} \Delta x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Delta u$$

$$\Delta y = \begin{pmatrix} 1 & 0 \end{pmatrix} \Delta x$$

# Solutions to Chapter 2. Dynamical Systems

**2.1 a.** The transfer function is

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B + D \\
&= \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} s+2 & 0 \\ 0 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + 2 \\
&= \frac{2s^2 + 7s + 1}{s^2 + 5s + 6}.
\end{aligned}
$$

From the transfer function it is easy to determine the differential equation

$$
\begin{aligned}
Y(s) &= G(s)U(s) \\
(s^2 + 5s + 6)Y(s) &= (2s^2 + 7s + 1)U(s) \\
\ddot{y} + 5\dot{y} + 6y &= 2\ddot{u} + 7\dot{u} + u
\end{aligned}
$$

**b.** The transfer function is

$$
\begin{aligned}
G(s) &= C(sI - A)^{-1}B + D \\
&= \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} s+7 & -2 \\ 15 & s-4 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \\
&= \frac{2s + 3}{s^2 + 3s + 2}.
\end{aligned}
$$

The differential equation becomes

$$
\begin{aligned}
Y(s) &= G(s)U(s) \\
(s^2 + 3s + 2)Y(s) &= (2s + 3)U(s) \\
\ddot{y} + 3\dot{y} + 2y &= 2\dot{u} + 3u
\end{aligned}
$$

**c.** $G(s) = \dfrac{5s + 8}{s + 1}, \quad \dot{y} + y = 5\dot{u} + 8u$

**d.** $G(s) = \dfrac{3s^2 + 7s + 18}{s^2 + 2s + 5}, \quad \ddot{y} + 2\dot{y} + 5y = 3\ddot{u} + 7\dot{u} + 18u$

**2.2 a.** Partial fraction expansion of the transfer function yields

$$
G(s) = 2 + \frac{2}{s+3} - \frac{5}{s+2}
$$

and by applying the inverse Laplace transform, one obtains the impulse response

$$
h(t) = \mathcal{L}^{-1}G(s) = 2\delta(t) + 2e^{-3t} - 5e^{-2t}, \quad t \geq 0.
$$

*Comment.* Because the system matrix was given in diagonal form, another possibility would have been to compute the impulse response as

$$h(t) = Ce^{At}B + D\delta(t) = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} + 2\delta(t), \quad t \geq 0.$$

The step response is computed by e.g. integrating the impulse response

$$\begin{aligned} y(t) &= \int_0^t h(\tau)d\tau = \int_0^t \left(2\delta(\tau) + 2e^{-3\tau} - 5e^{-2\tau}\right) d\tau \\ &= 2 + \left[\frac{5}{2}e^{-2\tau} - \frac{2}{3}e^{-3\tau}\right]_0^t \\ &= \frac{1}{6} + \frac{5}{2}e^{-2t} - \frac{2}{3}e^{-3t}, \quad t \geq 0. \end{aligned}$$

**b.** The transfer function has the partial fraction expansion

$$G(s) = \frac{1}{s+1} + \frac{1}{s+2}$$

and the impulse response becomes

$$h(t) = \mathcal{L}^{-1}G(s) = e^{-t} + e^{-2t}, \quad t \geq 0.$$

The step response is thus given by

$$y(t) = \int_0^t h(\tau)d\tau = \frac{3}{2} - e^{-t} - \frac{1}{2}e^{-2t}, \quad t \geq 0.$$

**c.** $h(t) = 5\delta(t) + 3e^{-t}, \quad y(t) = 8 - 3e^{-t}, \quad t \geq 0$

**d.** $h(t) = 3\delta(t) + e^{-t}\sin 2t + e^{-t}\cos 2t = 3\delta(t) + \sqrt{2}e^{-t}\sin\left(2t + \frac{\pi}{4}\right)$
$y(t) = 3 + \frac{1}{5}e^{-t}\left(3 + \sin 2t - 3\cos 2t\right), \quad t \geq 0$

**2.3** After the Laplace transform, one obtains

$$\begin{aligned} sX &= AX + BU \\ Y &= CX + DU \end{aligned}$$

Solve for $X$

$$\begin{aligned} (sI - A)X &= BU \\ X &= (sI - A)^{-1}BU \end{aligned}$$

This gives

$$Y = C(sI - A)^{-1}BU + DU = \left(C(sI - A)^{-1}B + D\right)U$$

**2.4 a.** The poles are the solutions of the characteristic equation $s^2+4s+3 = 0$, i.e. $s = -1$ and $s = -3$. The system lacks zeros.

**b.** The static gain is $G(0) = \frac{1}{3}$.

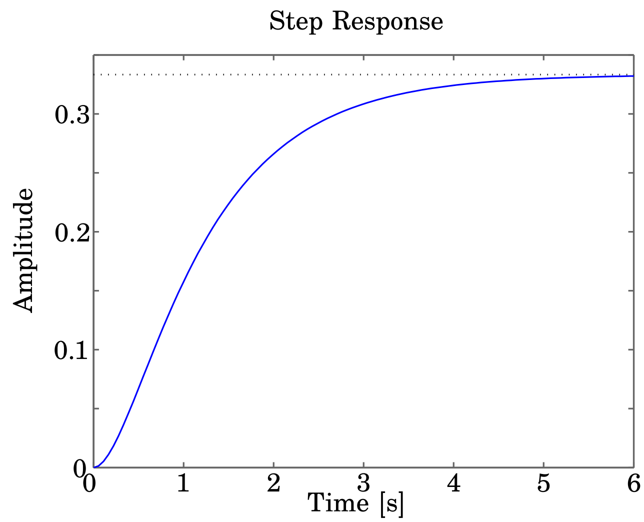
**c.** The input (a step) has the Laplace transform $U(s) = \frac{1}{s}$. The output becomes

$$Y(s) = G(s)U(s) = \frac{1}{s(s^2+4s+3)} = \frac{1}{s(s+1)(s+3)}$$

Inverse Laplace transformation (transform no. 24 with $a = 1$ and $b = 3$ gives

$$y(t) = \frac{1}{3} + \frac{e^{-3t} - 3e^{-t}}{6}$$

The step response is shown below.




**2.5 a.** The poles are the solutions the characteristic equation $s^2 + 0.6s + 0.25 = 0$, i.e. $s = -0.3 \pm 0.4i$. The system lacks zeros.

**b.** The static gain is $G(0) = 1$.

**c.** The input (a step) has the Laplace transform $U(s) = \frac{1}{s}$. The output becomes

$$Y(s) = G(s)U(s) = \frac{0.25}{s(s^2+0.6s+0.25)}$$

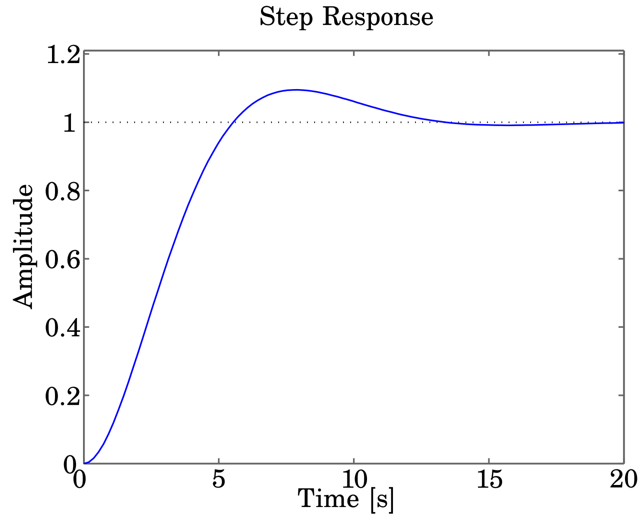
Because this system has complex poles, we first rewrite it as

$$Y(s) = \frac{\omega^2}{s(s^2+2\zeta\omega s+\omega^2)}$$

where $\omega = 0.5$ and $\zeta = 0.6$. We then utilize the inverse Laplace transformation (transform no. 28) and obtain

$$y(t) = 1 - 1.25e^{-0.3t}\sin(0.4t + 0.9273)$$

The step response is shown below.




**2.6** Laplace transformation of the differential equation $m\ddot{y}+c\dot{y}+ky=f$ yields

$$(ms^2+cs+k)Y=F$$

and the transfer function is hence

$$G(s)=\frac{1}{ms^2+cs+k}.$$

The poles are $s=-c/2m\pm i\sqrt{k/m-c^2/4m^2}$. A change in $k$ implies a change of the imaginary part of the poles. A change in $c$ affects both the real and imaginary parts.

The poles cannot end up in the right half plane due to physical reasons, since $c\geq 0$ and $m>0$.

**2.7 a.** $G(s)=\dfrac{1}{LCs^2+RCs+1}$

**b.** $G(s)=\dfrac{1}{Ts+1},\quad T=\dfrac{A}{a}\sqrt{\dfrac{2h^0}{g}}$

**2.8 a.** Initial value

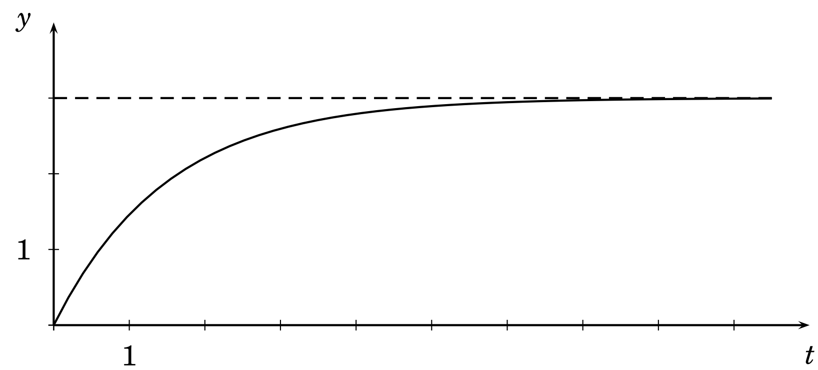
$$y(0)=\lim_{s\to+\infty}\frac{2}{s+2/3}=0.$$

Initial derivative

$$\dot{y}(0)=\lim_{s\to+\infty}\frac{2s}{s+2/3}=2.$$

Final value

$$\lim_{t\to+\infty}y(t)=G(0)=3.$$




**b.** Initial value

$$y(0) = \lim_{s \to +\infty} \frac{8}{s^2 + s + 4} = 0.$$

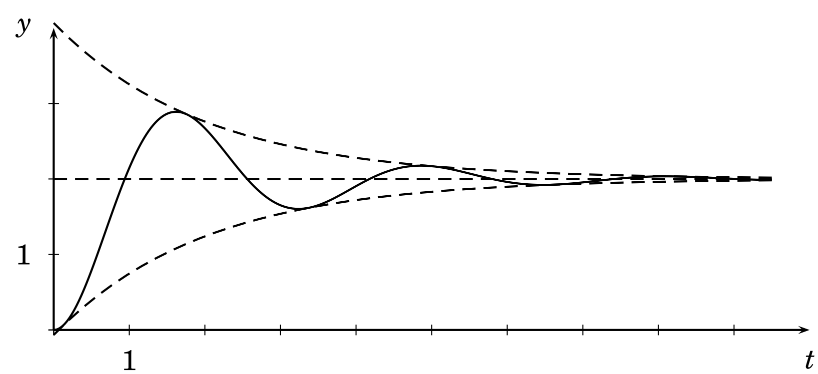
Initial derivative

$$\dot{y}(0) = \lim_{s \to +\infty} \frac{8s}{s^2 + s + 4} = 0.$$

Final value

$$\lim_{t \to +\infty} y(t) = G(0) = 2.$$

The transfer function has two complex poles $s = -1/2 \pm i\sqrt{15}/2$ and thus the step response should oscillate with period $T = 4\pi/\sqrt{15} \approx 3$ and damping $\sigma = 1/2$.



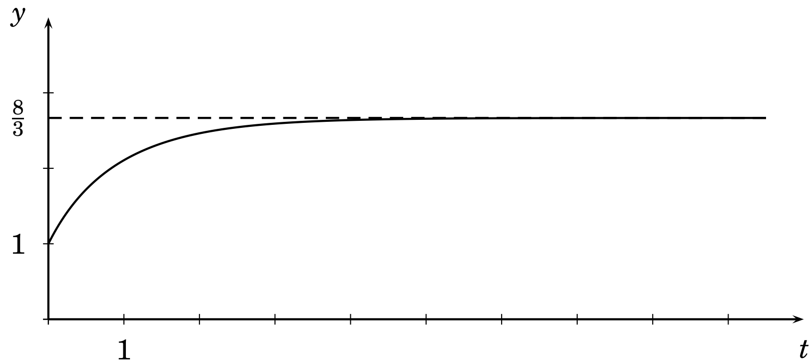

**c.** The transfer function has poles in $-1$ and $-3$ and zeros in $-2$ and $-4$. Polynomial division and partial fraction expansion yields

$$G(s) = 1 + \frac{3}{2s + 2} + \frac{1}{2s + 6}.$$

The initial value is $y(0) = \lim_{s \to +\infty} G(s) = 1$ and the initial derivative becomes

$$\dot{y}(0) = \lim_{s \to +\infty} \left( \frac{3s}{2s + 2} + \frac{s}{2s + 6} \right) = 2.$$

The final value is $\lim_{t \to +\infty} y(t) = G(0) = 8/3$.




**2.9**

| | $y(0)$ | $\dot{y}(0)$ | final value | poles | period | step response |
|---|---|---|---|---|---|---|
| $G_1$ | 0 | 0.1 | 1 | $-0.1$ | — | — |
| $G_2$ | 0 | 0 | 1 | $-1 \pm i\sqrt{3}$ | 3.6 | A |
| $G_3$ | 0 | 0 | — | $0.05 \pm i\sqrt{2.00}$ | 4.4 | E |
| $G_4$ | 0 | 0 | $-0.25$ | $-0.05 \pm i\sqrt{2.00}$ | 4.4 | — |
| $G_5$ | 0 | 1 | 1 | $-1$ | — | D |
| $G_6$ | 0 | 0 | 1 | $-0.4 \pm i\sqrt{3.84}$ | 3.2 | B |
| $G_7$ | 0 | 0 | 0.67 | $-0.5 \pm i\sqrt{2.75}$ | 3.8 | C |

**2.10** $\boxed{1}$ The system has the poles in $-1/4 \pm i$ and a zero in $-1$. The transfer function is thus

$$G(s) = K\frac{s+1}{(s+\frac{1}{4})^2+1} \approx K\frac{s+1}{s^2+\frac{1}{2}s+1}.$$

The initial value, initial derivative and final value become

$$\begin{aligned} y(0) &= \lim_{s\to+\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to+\infty} sG(s) = K \neq 0 \\ \lim_{t\to+\infty} y(t) &= G(0) = K \neq 0 \end{aligned}$$

The step response is oscillating with period $T = 2\pi/1 \approx 6$. This must be step response D.

$\boxed{2}$ The system has poles in $-1$ and $-2$ and a zero in 1. The transfer function is

$$G(s) = K\frac{s-1}{(s+1)(s+2)}$$

The initial value, initial derivative and final value become

$$\begin{aligned} y(0) &= \lim_{s\to+\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to+\infty} sG(s) = K \neq 0 \\ \lim_{t\to+\infty} y(t) &= G(0) = -\frac{K}{2} \neq 0 \end{aligned}$$

We see that the initial derivative and the final value have different signs. This fits step response F.

[3] The system has poles in $-1/4 \pm i$ and a zero in 0. The transfer function is

$$G(s) = K\frac{s}{(s+\frac{1}{4})^2+1} \approx K\frac{s}{s^2+\frac{1}{2}s+1}$$

The initial value, initial derivative and final value become

$$\begin{aligned} y(0) &= \lim_{s\to+\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to+\infty} sG(s) = K \neq 0 \\ \lim_{t\to+\infty} y(t) &= G(0) = 0 \end{aligned}$$

The step response is oscillating with period $T = 2\pi/1 \approx 6$. This is step response G.

[4] The system has poles in $-1$ and $-2$ and a zero in $-3$. The transfer function is

$$G(s) = K\frac{s+3}{(s+1)(s+2)}.$$

The initial value, initial derivative and final value become

$$\begin{aligned} y(0) &= \lim_{s\to+\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to+\infty} sG(s) = K \neq 0 \\ \lim_{t\to+\infty} y(t) &= G(0) = \frac{3K}{2} \neq 0 \end{aligned}$$

The initial derivative and final value have the same sign. The only nonoscillative step response which suits these criteria is C.

**2.11a.**

$$\begin{aligned} Y &= G_1(U + G_2Y) \\ Y(1 - G_1G_2) &= G_1U \\ Y &= \frac{G_1}{1 - G_1G_2}U \end{aligned}$$

**b.**

$$\begin{aligned} Y &= G_2(H_1U + G_1U + H_2Y) \\ Y(1 - G_2H_2) &= (G_2H_1 + G_2G_1)U \\ Y &= \frac{G_2H_1 + G_2G_1}{1 - G_2H_2}U \end{aligned}$$

**c.** Introduce the auxiliary variable $Z$, being the output of $G_1$

$$\begin{aligned} Z &= G_1(U + G_3(Z + G_2Z)) \\ Z(1 - G_1G_3 - G_1G_3G_2) &= G_1U \\ Z &= \frac{G_1}{1 - G_1G_3 - G_1G_3G_2}U \\ Y &= \frac{G_2G_1}{1 - G_1G_3 - G_1G_3G_2}U \end{aligned}$$

**d.**

$$\begin{aligned} Y &= G_2(-H_2Y + G_1(U - H_1Y)) \\ Y(1 + G_2H_2 + G_2G_1H_1) &= G_2G_1U \\ Y &= \frac{G_2G_1}{1 + G_2H_2 + G_2G_1H_1}U \end{aligned}$$

**2.12a.** Partial fraction expansion yields

$$G(s) = \frac{s^2 + 6s + 7}{s^2 + 5s + 6} = \frac{s+1}{s^2 + 5s + 6} + 1 = \frac{-1}{s+2} + \frac{2}{s+3} + 1$$

One has a certain freedom when choosing the coefficients of the $B$ and $C$ matrices. However, the products $b_ic_i$ remain constant. Let e.g. $b_1 = b_2 = 1$. This enables us to immediately write the system in diagonal form:

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} -1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \end{pmatrix} u \end{aligned}$$

**b.** First rewrite the system as

$$G(s) = \frac{b_0s + b_1}{s^2 + a_1s + a_2} + d = \frac{s+1}{s^2 + 5s + 6} + 1$$

The controllable canonical form can be directly read from the transfer function

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -5 & -6 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \end{pmatrix} u \end{aligned}$$

**c.** The observable canonical form is obtained in the same manner

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} -5 & 1 \\ -6 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \end{pmatrix} u \end{aligned}$$

# Solutions to Chapter 3. Frequency Analysis

**3.1 a.** The output is given by

$$y(t) = |G(3i)| \sin\Big(3t + \arg G(3i)\Big)$$

where

$$|G(i\omega)| = \frac{0.01\sqrt{1+100\omega^2}}{\sqrt{1+\omega^2}\sqrt{1+0.01\omega^2}}$$

and

$$\arg G(i\omega) = \arctan 10\omega - \arctan \omega - \arctan 0.1\omega$$

For $\omega = 3$ one obtains $|G(i\omega)| = 0.0909$ and $\arg G(i\omega) = -0.003$ which gives

$$y(t) = 0.0909 \sin(3t - 0.003)$$

**b.** Reading from the plot yields $|G(3i)| \approx 0.09$ and $\arg G(3i) \approx 0$. We obtain

$$y(t) = 0.09 \sin 3t$$

**3.2 a.** The output is given by

$$y(t) = |G(i\omega)| \sin\Big(\omega t + \arg G(i\omega)\Big)$$

where

$$|G(i\omega)| = \left|\frac{10}{(i\omega)^2 + 0.5i\omega + 1}\right| = \frac{10}{\sqrt{(1-\omega^2)^2 + (0.5\omega)^2}}$$

and

$$\arg G(i\omega) = \arg \frac{10}{(i\omega)^2 + 0.5i\omega + 1} = -\arg\big((1-\omega^2) + 0.5\omega i\big)$$

$$= \begin{cases} -\arctan \dfrac{0.5\omega}{1-\omega^2}, & \omega < 1 \\ -\pi/2, & \omega = 1 \\ -\arctan \dfrac{0.5\omega}{1-\omega^2} - \pi, & \omega > 1 \end{cases}$$

The output becomes

$$10.4 \sin(0.2t - 5.9^\circ), \quad 20.0 \sin(t - 90.0^\circ), \quad 0.011 \sin(30t - 179.0^\circ)$$

**b.** For $\omega = 0.2$ one reads $|G(i\omega)| \approx 10$ and $\arg G(i\omega) \approx -5^\circ$. For $\omega = 1$ one reads $|G(i\omega)| \approx 20$ and $\arg G(i\omega) \approx -90^\circ$. For $\omega = 30$ one reads $|G(i\omega)| \approx 0.01$ and $\arg G(i\omega) \approx -180^\circ$. The output is approximately

$$10 \sin(0.2t - 5^\circ), \quad 20 \sin(t - 90^\circ), \quad 0.01 \sin(30t - 180^\circ)$$

**3.3** We use the following general approach to draw Bode plots

- Factor the transfer function of the system.
- Determine the low frequency asymptote (small $s$).
- Determine the corner frequencies (i.e. the magnitude of the poles and zeros of the system.)
- Draw the asymptotes of the gain curve from low to high frequencies, aided by the following rules of thumb
  - A pole decreases the slope of the gain curve by 1 at the corner frequency.
  - A zero increases the slope of the gain curve by 1 at the corner frequency.
- Draw the asymptotes of the phase curve from low to high frequencies, aided by the following rules of thumb
  - A (stable) pole decreases the value of the phase curve by 90° at the corner frequency.
  - A (stable) zero increases the value of the phase curve by 90° at the corner frequency.
- Draw the real gain- and phase curves, aided by the asymptotes and sample curves in the collection of formulae.

**a.** The transfer function can be written

$$G(s) = 3 \cdot \frac{1}{1 + s/10}$$

Low frequency asymptote: $G(s) \approx 3$.

Corner frequency: $\omega = 10$ rad/s (pole).

The gain curve starts with slope 0 and value 3. The slope decreases by 1 at $\omega = 10$ rad/s, due to the pole, and thus ends being $-1$.

The phase curve starts at 0°. The phase is decreased by 90° at $\omega = 10$ rad/s, due to the pole, and thus ends being $-90°$.

The asymptotes and the finished Bode plots are shown in figure 3.1.

**b.** The transfer function can be written

$$G(s) = 10 \cdot \frac{1}{1 + 10s} \cdot \frac{1}{1 + s}$$

Low frequency asymptote: $G(s) \approx 10$.

Corner frequencies: $\omega = 0.1$ rad/s (pole), $\omega = 1$ rad/s (pole).

The gain curve starts with slope 0 and value 10. The slope is decreased by 1 at $\omega = 0.1$ rad/s, due to the first pole, and by 1 at $\omega = 1$ rad/s, due to the second pole. Thus, the final slope becomes $-2$.

The phase curve starts at 0°. The phase is decreased by 90° at $\omega = 0.1$ rad/s, due to the first pole, and by 90° at $\omega = 1$ rad/s, due to the second pole. Thus, the final phase is $-180°$.

The asymptotes and the finished Bode plot are shown in figure 3.2.

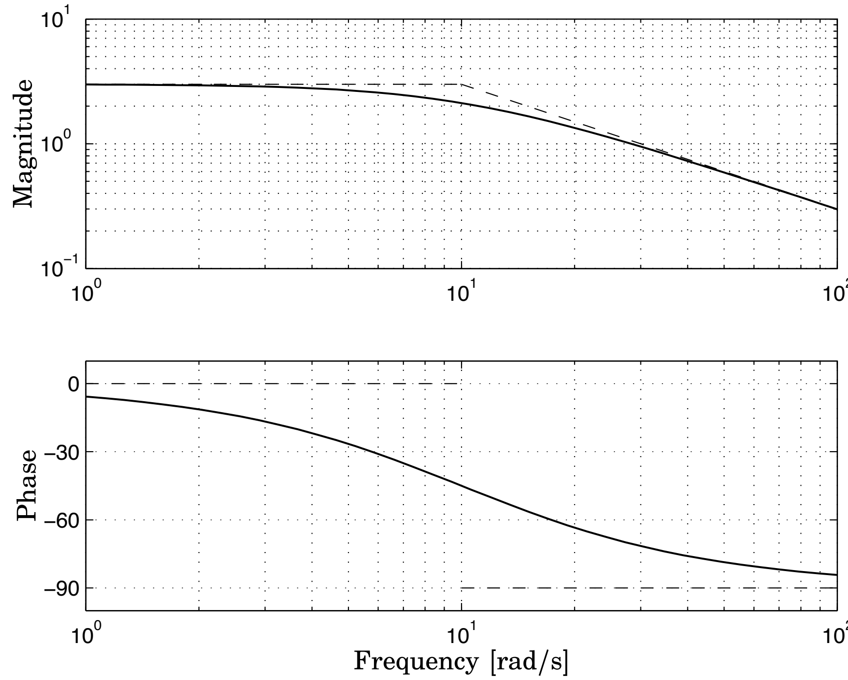



**Figure 3.1** The Bode plot of $G(s) = \frac{3}{1+s/10}$.

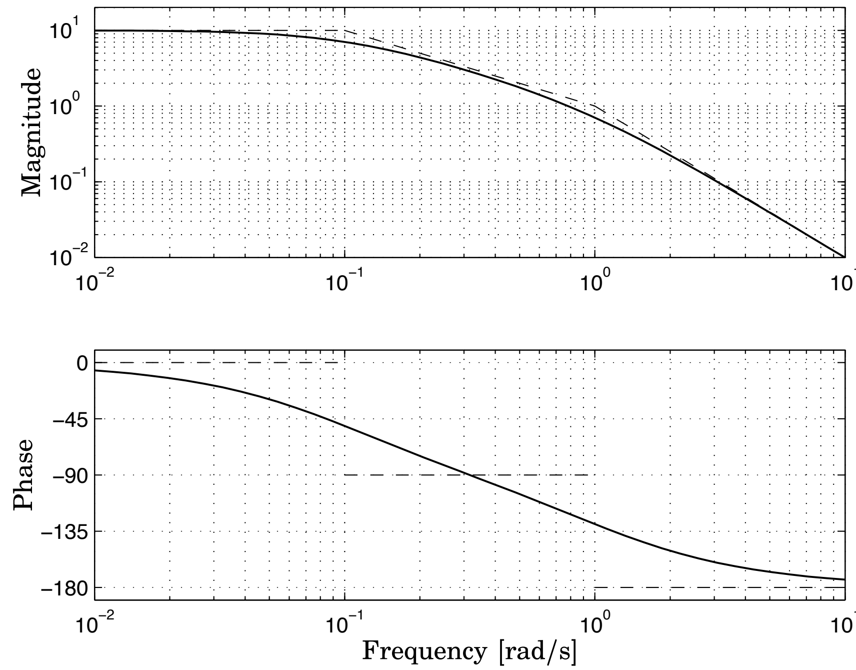



**Figure 3.2** Bode plot of $G(s) = \frac{10}{(1+10s)(1+s)}$.

**c.** The transfer function can be written

$$G(s) = e^{-s} \cdot \frac{1}{1+s}$$

Low frequency asymptote: $G(s) \approx 1$.
Corner frequency: $\omega = 1$ rad/s (pole).

The delay ($e^{-s}$) does not affect the gain curve, which starts with slope 0 and value 1. The slope is decreased by 1 at $\omega = 1$ rad/s, due to the pole, and the final slope is thus $-1$.

The phase curve is harder to sketch. One approach is to draw the asymptotes of the system without the delay and superposition it with the phase curve of $e^{-s}$, which can be obtained from the collection of formulae or by computing some points and interpolating between these.

Anyway, we see that the phase curve starts at 0° and that the phase then decreases both due to the pole (at $\omega = 1$ rad/s) and the delay. The delay causes the phase to approach $-\infty$ for large $\omega$.

The finished plot is shown in figure 3.3.

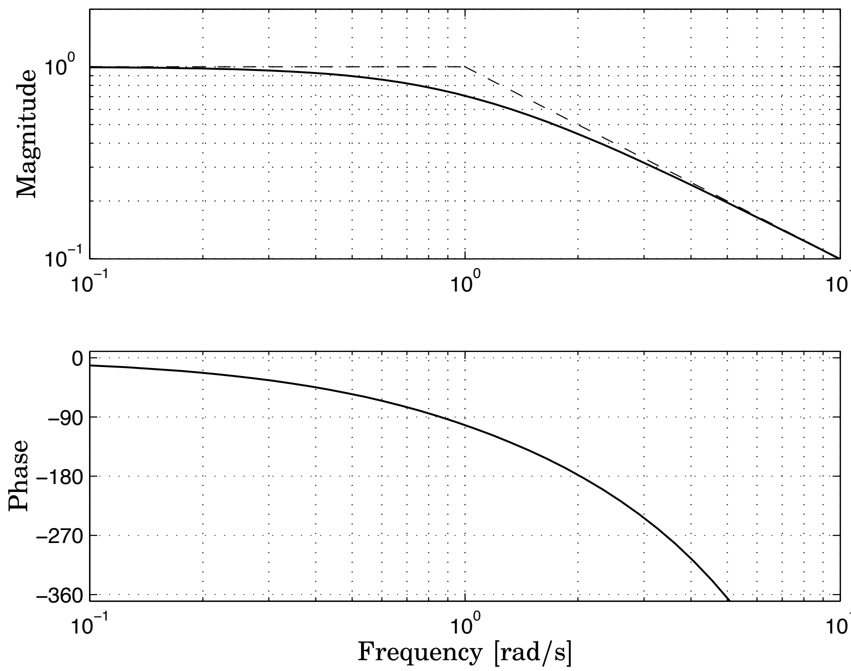



**Figure 3.3** Bode plot of $G(s) = \frac{e^{-s}}{1+s}$.

**d.** The transfer function can be written

$$G(s) = \frac{1}{s} \cdot (1+s) \cdot \frac{1}{1+s/10}$$

Low frequency asymptote: $G(s) \approx \dfrac{1}{s}$.

Corner frequencies: $\omega = 1$ rad/s (zero), $\omega = 10$ rad/s (pole).

The gain curve starts with slope $-1$. The slope increases by 1 at $\omega = 1$ rad/s, due to the zero, and at $\omega = 10$ rad/s the slope decreases by 1, due to the pole. Consequently, the final slope is $-1$.

The phase curve starts at $-90°$. The phase increases by 90° at $\omega = 1$ rad/s, due to the zero, and decreases by 90° at $\omega = 10$ rad/s, due to the pole. Consequently, the final phase is $-90°$.

The finished plot is shown in figure 3.4.

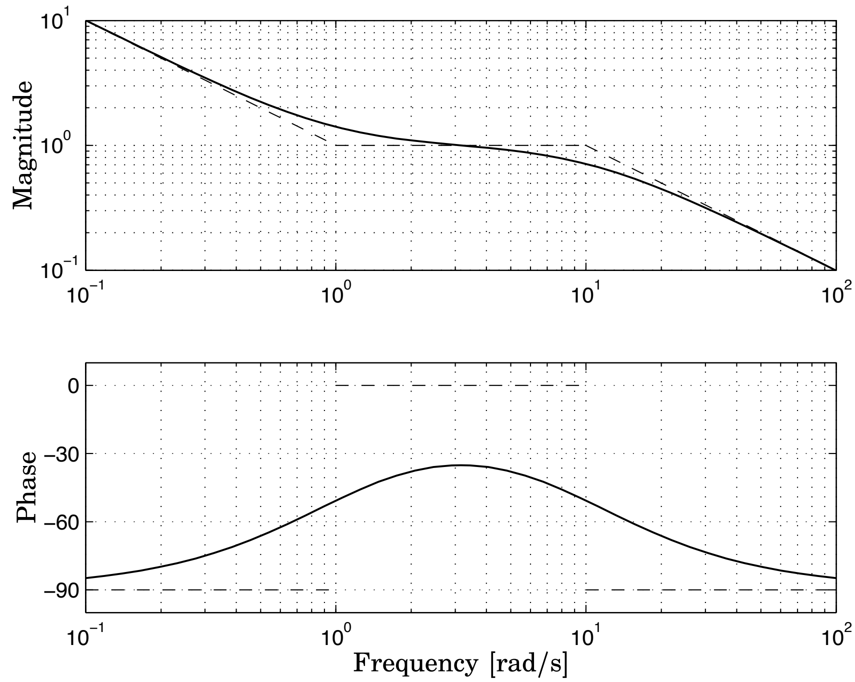



**Figure 3.4** Bode plot of $G(s) = \frac{1+s}{s(1+s/10)}$.

**e.** The transfer function can be written

$$G(s) = 2 \cdot \frac{1}{s} \cdot (1 + 5s) \cdot \frac{1}{1 + 2\zeta(s/2) + (s/2)^2}$$

where $\zeta = 0.2$.

Low frequency asymptote: $G(s) \approx \frac{2}{s}$.

Corner frequencies: $\omega = 0.2$ rad/s (zero), $\omega = 2$ rad/s (complex conjugated pole pair).

The gain curve starts with slope $-1$. The slope is increased by 1 at $\omega = 0.2$ rad/s, due to the zero, and decreased by 2 at $\omega = 2$ rad/s, due to the pole pair. Consequently, the final slope is $-2$.

The phase curve starts at $-90°$. The phase is increased by $90°$ at $\omega = 0.2$ rad/s, due to the zero, and decreased by $180°$ at $\omega = 2$ rad/s, due to the pole pair. Consequently, the final phase is $-180°$.

The low damping ($\zeta = 0.2$) of the complex conjugated pole pair gives the gain curve a resonance peak at $\omega = 2$ rad/s. Additionally, the phase decreases rapidly at this frequency, cf. the sample curves in the collection of formulae. The finished plot is shown in figure 3.5.

**3.4 a.** The Nyquist curve start in 3 (the static gain) for $\omega = 0$ rad/s. Both the gain and phase are strictly decreasing, which makes the curve turn clockwise while its distance to the origin decreases. The gain and phase approach 0 and $-90°$, respectively, for large values of $\omega$. The curve is thus bound to the fourth quadrant and approaches the origin along the negative imaginary axis as $\omega \to \infty$.

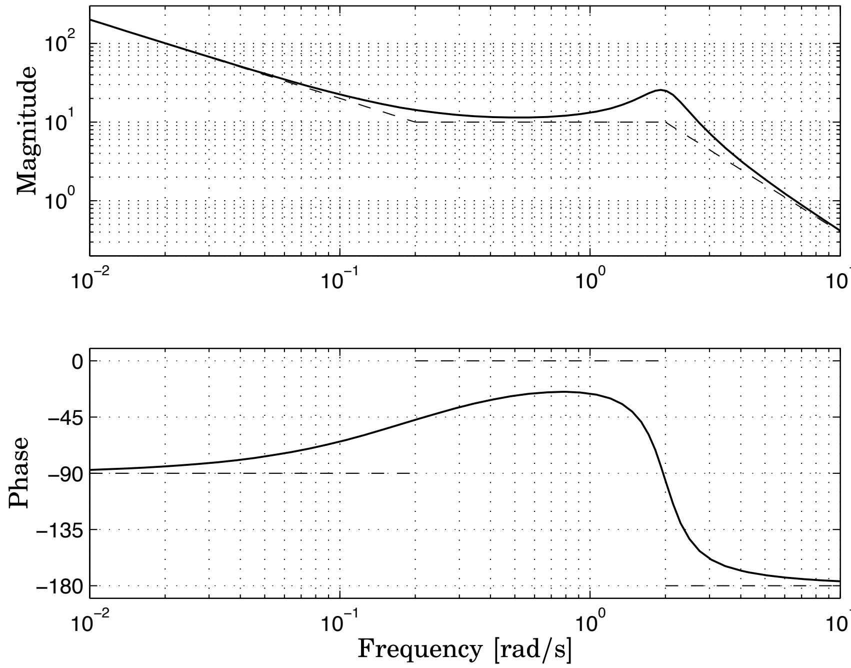



**Figure 3.5** Bode plot of $G(s) = \frac{2(1+5s)}{s(1+0.2s+0.25s^2)}$.

Aided by this analysis, one can now sketch the Nyquist curve by choosing a few frequencies (e.g. $\omega = 1$, 10 and 100 rad/s) and drawing the corresponding points in the complex plane. The finished curve is shown in figure 3.6.

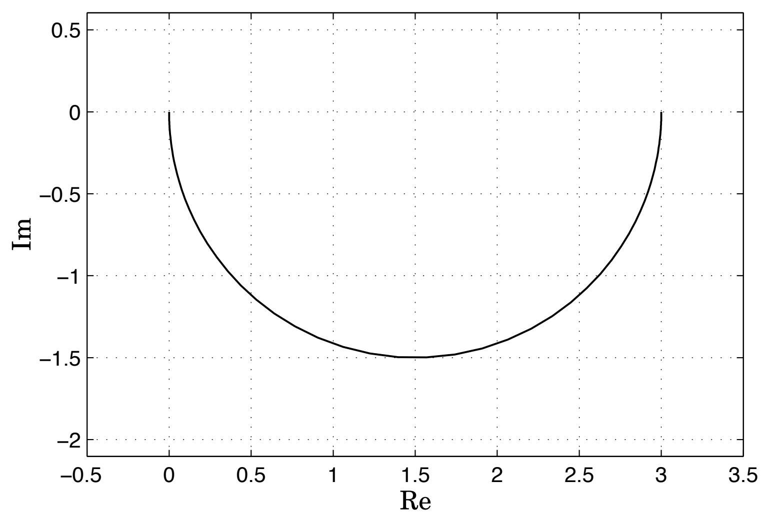



**Figure 3.6** Nyquist curve of $G(s) = \frac{3}{1+s/10}$.

**b.** The Nyquist curve starts in 10 (the static gain) for $\omega = 0$ rad/s. Both the gain and phase are strictly decreasing, which makes the curve turn clockwise while its distance to the origin decreases. The gain and phase approach 0 and $-180°$, respectively, for large values of $\omega$. The curve will thus go from the fourth to the third quadrant, approaching the origin along the negative real axis as $\omega \to \infty$.

The intersection with the negative imaginary axis can be drawn by reading off the magnitude when the phase is $-90°$. One can now sketch the Nyquist curve by choosing a few additional frequencies (e.g. $\omega = 0.1$, 1 rad/s) and drawing the corresponding points in the complex plane. The finished curve is shown in figure 3.7.

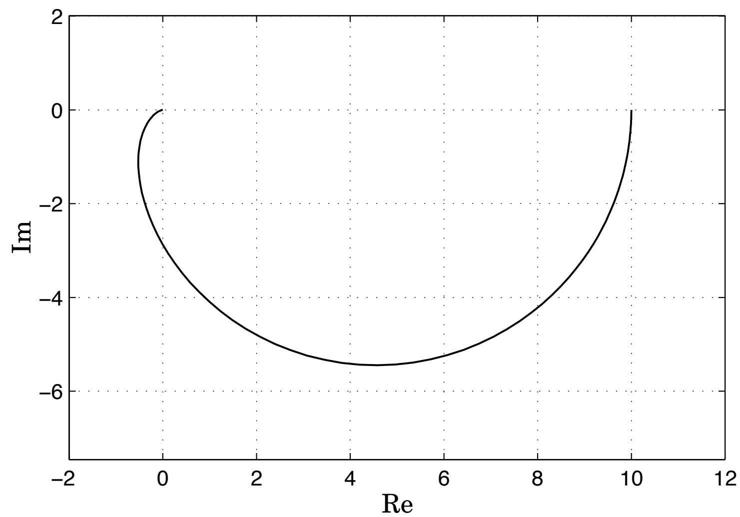



**Figure 3.7** Nyquist curve of $G(s) = \frac{10}{(1+10s)(1+s)}$.

**c.** The Nyquist curve starts in 1 (the static gain) for $\omega = 0$ rad/s. Both the gain and phase are strictly decreasing, which makes the curve turn clockwise while its distance to the origin decreases. The gain and phase approach 0 and $-\infty$, respectively, for large values of $\omega$. The curve will thus rotate infinitely many times as it approaches the origin. The first intersections with the axis can be drawn by reading off the magnitude when the phase is $-90°$, $-180°$, $-270°$ and $-360°$, respectively. The finished curve is shown in figure 3.8.

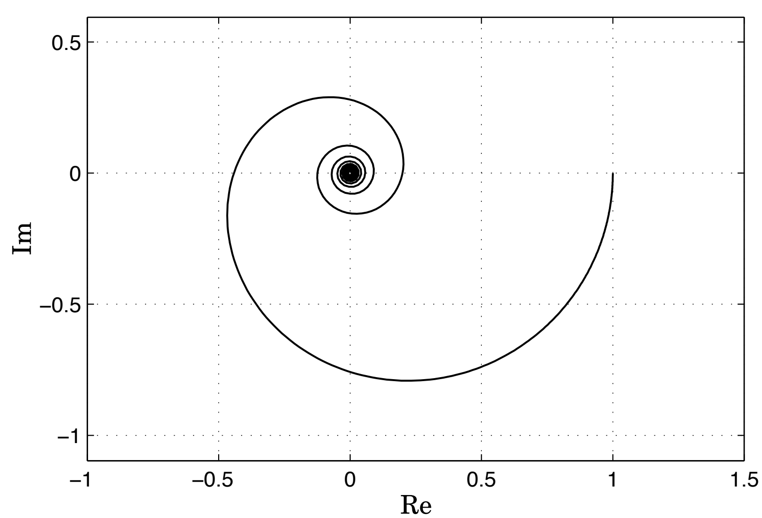



**Figure 3.8** Nyquist curve of $G(s) = \frac{e^{-s}}{1+s}$.

**3.5** Let the sought transfer function be $G(s)$. The gain curve starts with slope $-1$, which indicates that $G(s)$ contains a factor $\frac{1}{s}$ (an integrator). We observe that there are two corner frequencies: $\omega_1 = 1$ and $\omega_2 = 100$ rad/s. The gain curve breaks upwards once at $\omega_1$ and downward once at $\omega_2$. Hence, the nominator hosts a factor $1+s$, whereas the denominator contains a factor $1+s/100$. In addition, $G(s)$ contains a constant gain $K$. We thus have

$$G(s) = \frac{K(1+s)}{s(1+s/100)}$$

We evaluate the low frequency asymptote of the gain curve at e.g. $\omega = 0.01$ rad/s, in order to determine $K$. This yields

$$|G(0.01i)| = \frac{K}{0.01} = 1 \quad \Rightarrow \quad K = 0.01$$

Finally we verify that the phase curve matches this system.

**3.6** Let the sought transfer function be $G(s)$. The gain curve has two corner frequencies: $\omega_1 = 2$ and $\omega_2 = 100$ rad/s. The gain curve breaks downwards once at $\omega_1$ and three times at $\omega_2$. Thus the denominator of $G(s)$ contains the factors $(1+\frac{s}{2})$ and $(1+\frac{s}{100})^3$. The slope of the low frequency asymptote is 1. Thus $G(s)$ has a factor $s$ in the nominator. Additionally, $G(s)$ contains a constant gain $K$. We have

$$G(s) = \frac{Ks}{(1+\frac{s}{2})(1+\frac{s}{100})^3}$$

The factor $K$ is computed by determining a point on the LF asymptote, e.g. $G_{LF}(s) = Ks$

$$|G_{LF}(i\omega)| = K\omega = 4$$

for $\omega = 2$ rad/s. This gives

$$K = 2$$

(Observe that one should use the LF asymptote rather than the actual gain curve, when computing $K$.)

Finally we verify by checking that the phase curve matches this system.

# Solutions to Chapter 4. Feedback Systems

**4.1 a.** Laplace transformation of the differential equation yields

$$sY(s) + 0.01Y(s) = 0.01U(s)$$

The transfer function $G_P(s)$ is thus given by

$$Y(s) = G_P(s)U(s) = \frac{0.01}{s + 0.01}U(s)$$

**b.** The transfer function of the closed loop system becomes

$$G(s) = \frac{G_P(s)G_R(s)}{1 + G_P(s)G_R(s)} = \frac{\frac{0.01}{s+0.01}K}{1 + \frac{0.01}{s+0.01}K} = \frac{0.01K}{s + 0.01 + 0.01K}$$

**c.** The desired and actual characteristic polynomials are the same if all their coefficients match. Identification of coefficients yields

$$0.1 = 0.01 + 0.01K \quad \Leftrightarrow \quad K = 9$$

**4.2** Since $r(t) = 0$, the control error becomes $e(t) = -y(t)$. Further, the closed loop system has to be asymptotically stable.

$$Y(s) = G_P(s)(F(s) - G_R(s)Y(s)) \quad \Leftrightarrow \quad Y(s) = \frac{G_P(s)}{1 + G_R(s)G_P(s)}F(s)$$

If $f(t)$ is a unit step, we have $F(s) = \frac{1}{s}$.

**a.** Seek $y(\infty)$ for $G_R = K$

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{1}{(ms^2 + ds + K)}\frac{1}{s} = \frac{1}{K}$$

The function $sY(s)$ has all poles in the left-half plane when the parameters $m$, $d$ and $K$ are positive.

**b.** The same assignment, but with $G_R(s) = K_1 + K_2/s$. This yields

$$\begin{aligned} y(\infty) &= \lim_{s\to 0} s\frac{1}{(ms^2 + ds + K_1 + \frac{K_2}{s})}\frac{1}{s} \\ &= \lim_{s\to 0} \frac{s}{ms^3 + ds^2 + K_1 s + K_2} = 0 \end{aligned}$$

under the assumption of stability, which is the case for $m > 0$, $d > 0$ and $K_1 > \frac{m}{d}K_2 > 0$. Rule: If the disturbance is a step, one needs at least one integrator before the point in the block diagram where the disturbance is introduced, in order to make the stationary error zero.

**4.3 a.** For the closed loop system it holds, when $R = 0$, that

$$U(s) = K(0 - Y(s)) = -K(G_P(s)U(s) + N(s))$$

from which one obtains

$$\begin{aligned} U(s) &= \frac{-K}{1 + KG_P(s)} N(s) \\ Y(s) &= G_P(s)U(s) + N(s) = \frac{1}{1 + KG_P(s)} N(s) \end{aligned} \tag{4.1}$$

**b.** Inserting $G_P(s) = \frac{1}{s+1}$ into (4.1) yields the relations

$$\begin{aligned} U(s) &= \frac{-K(s+1)}{s+1+K} N(s) \\ Y(s) &= G_P(s)U(s) + N(s) = \frac{s+1}{s+1+K} N(s) =: G_{yn}(s)N(s) \end{aligned}$$

In stationarity it holds that

$$\begin{aligned} y(t) &= A|G_{yn}(i\omega)| \sin(\omega t + \arg G_{yn}(i\omega)) \\ &= A \frac{\sqrt{1+\omega^2}}{\sqrt{(K+1)^2 + \omega^2}} \sin\left(\omega t + \arctan \omega - \arctan \frac{\omega}{K+1}\right) \\ u(t) &= -Ky(t) \\ &= -KA \frac{\sqrt{1+\omega^2}}{\sqrt{(K+1)^2 + \omega^2}} \sin\left(\omega t + \arctan \omega - \arctan \frac{\omega}{K+1}\right) \end{aligned}$$

**c.** With $A = 1$ and $K = 1$ the amplitudes of the oscillations in $u$ and $y$ become

$$\begin{aligned} A_u &= \sqrt{\frac{1+\omega^2}{4+\omega^2}} \\ A_y &= \sqrt{\frac{1+\omega^2}{4+\omega^2}} \end{aligned}$$

For $\omega = 0.1$ rad/s the amplitudes become

$$\begin{aligned} A_u &\approx 0.5 \\ A_y &\approx 0.5 \end{aligned}$$

while $\omega = 10$ rad/s yields

$$\begin{aligned} A_u &\approx 1 \\ A_y &\approx 1 \end{aligned}$$

**4.4** With $G_P(s) = 1/(Js^2)$ we obtain

$$\begin{aligned} E(s) &= \theta_{ref}(s) - \theta(s) \\ &= \theta_{ref}(s) - G_P(s)(M(s) + KG_R(s)E(s)) \\ \Rightarrow E(s) &= \frac{1}{1 + KG_P(s)G_R(s)} \theta_{ref}(s) - \frac{G_P(s)}{1 + KG_P(s)G_R(s)} M(s) \end{aligned}$$

Assume a constant disturbance momentum $M_d^0$ and a constant reference $\theta_{ref}^0$. We postulate $G_R(s) = Q(s)/P(S)$, which gives

$$E(s) = \frac{1}{1 + K\dfrac{Q(s)}{Js^2P(s)}} \cdot \frac{\theta_{ref}^0}{s} - \frac{\dfrac{1}{Js^2}}{1 + K\dfrac{Q(s)}{Js^2P(s)}} \cdot \frac{M_d^0}{s}$$

$$= \frac{s^2JP(s)}{s^2JP(s) + KQ(s)} \cdot \frac{\theta_{ref}^0}{s} - \frac{P(s)}{s^2JP(s) + KQ(s)} \cdot \frac{M_d^0}{s}$$

The stationary error becomes

$$e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= 0 - \frac{P(0)}{KQ(0)}M_d^0 = -\frac{P(0)}{KQ(0)}M_d^0$$

where we have assumed that $Q(0) \neq 0$ and that the conditions for the final value theorem are fulfilled. We see that $P(0) = 0$ yields $e_\infty = 0$. In order to eliminate persistent angular errors caused by disturbance momenta, it is consequently required to utilize a controller $G_R(s)$ with at least one pole in the origin ($P(0) = 0$).

**4.5** The input of the thermocouple is the temperature $u(t)$ of the bath, which gives

$$u(t) = t \quad \Rightarrow \quad U(s) = \frac{1}{s^2}$$

The output $y(t)$ is the reading of the temperature sensor. Thus

$$Y(s) = G(s)U(s) = \frac{1}{1+sT} \cdot \frac{1}{s^2}$$

For the error $e(t) = u(t) - y(t)$ it holds that

$$E(s) = U(s) - Y(s) = \frac{1}{s^2}\left[1 - \frac{1}{1+sT}\right] = \frac{sT}{1+sT} \cdot \frac{1}{s^2}$$

The stationary error is obtained by means of the final value theorem

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^2T}{1+sT}\frac{1}{s^2} = T = 10$$

The thermocouple measurement is hence 10° less than the actual temperature. I.e. the actual temperature of the bath is 102.6°C + 10°C = 112.6°C.

Observe that the error in this has a bounded limit, despite the fact that both $u(t)$ and $y(t)$ lack (bounded) limits as $t \to \infty$. It is the *difference* between $u$ and $y$ which converges to a constant value.

**4.6** The low frequency asymptote is

$$G_{LF}(s) = \frac{K}{s^2}$$

where the constant K is given by

$$|G_{LF}(i\omega)| = \frac{K}{\omega^2}; \quad |G_{LF}(i)| = 1 \quad \Rightarrow \quad K = 1$$

At the corner frequency $\omega_1 = 1$ rad/s the slope changes from $-2$ to 0, and at $\omega_2 = 5$ rad/s it changes from 0 to $-1$. The transfer function for the open loop system is thus

$$G_o(s) = \frac{(1+sT_1)^2}{s^2(1+sT_2)}$$

where $T_1 = 1/\omega_1 = 1$ and $T_2 = 1/\omega_2 = 0.2$.
The transfer function of the closed loop system becomes

$$G(s) = \frac{G_o(s)}{1+G_o(s)}$$

The output is $Y(s) = G(s)R(s)$ and the error $E(s)$ becomes

$$E(s) = R(s) - Y(s) = \frac{1}{1+G_o}R(s) = \frac{s^2(1+0.2s)}{s^2(1+0.2s)+(1+s)^2}R(s)$$

**a.**

$$R(s) = \frac{a}{s} \quad \Rightarrow \quad e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{as^2(1+0.2s)}{s^2(1+0.2s)+(1+s)^2} = 0$$

The system can thus track inputs $r(t) = a$ without a stationary error.

**b.**

$$R(s) = \frac{b}{s^2} \quad \Rightarrow \quad e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{bs(1+0.2s)}{s^2(1+0.2s)+(1+s)^2} = 0$$

The system can also track inputs $r(t) = bt$ without a stationary error.

**c.**

$$R(s) = \frac{2c}{s^3} \quad \Rightarrow \quad e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{2c(1+0.2s)}{s^2(1+0.2s)+(1+s)^2} = 2c \neq 0$$

The input $r(t) = ct^2$, however, yields a stationary error.

**d.** Superposition can be used, since the closed loop system is linear and time invariant (LTI). Here, the input is the sum of the inputs in sub-assignments a and b. The total (superpositioned) stationary error thus becomes $e_\infty = 0 + 0 = 0$.

**e.** The input $r(t) = \sin(t)$ yields

$$R(s) = \frac{1}{1+s^2} \quad \Rightarrow \quad \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s \frac{s^2(1+0.2s)}{(s^2(1+0.2s)+(1+s)^2)(1+s^2)} = 0$$

but the input $r(t) = \sin(t)$ yields the output $y(t) = y_o \sin(t+\phi)$, where

$$y_o = |G(i)|\,, \quad \phi = arg\ G(i)$$

once transients have decayed. The error $e(t) = r(t) - y(t)$ is thus also a sinusoid and the limit

$$\lim_{t\to\infty} e(t)$$

does not exist. This shows that the final value theorem should not be used without caution. It is only valid for cases where a limit really exists. The criterion is that all poles of $sE(s)$ must have negative real parts. (The factor $s^2+1$ in the denominator yields two poles on the imaginary axis.)

**4.7 a.** The sensitivity function is given by

$$S(s) = \frac{1}{1+G_P(s)G_R(s)} = \frac{1}{1+\frac{6.5}{(s+1)^3}} = \frac{s^3+3s^2+3s+1}{s^3+3s^2+3s+7.5}$$

**b.** For $\omega = 0$ rad/s we have $|S(i\omega)| = 1/7.5$. Constant load disturbances are thus damped by a factor 7.5. The sensitivity functions has its maximum value $|S(i\omega)| =\approx 10$ at $\omega \approx 1.6$ rad/s.

**4.8 a.** The left curve shows the complementary sensitivity function, whereas the sensitivity function is given by the right curve.

**b.** The disturbances at various frequencies are amplified according to the gain curve of the sensitivity function. Disturbances below 2 rad/s are hence reduced, disturbances in the range 0.2 to 2 rad/s are amplified and disturbances above 2 rad/s pass straight through. The worst case gain, 2, is obtained at the frequency 0.55 rad/s.

**c.** The complementary sensitivity function, corresponding to the closed loop transfer function from $r$ to $y$, lies close to 1 up to approximately 0.7 rad/s.

**d.** The maximal magnitude of the sensitivity function equals the inverse of the minimal distance between the Nyquist curve and the point $-1$. The minimal distance is thus $1/2 = 0.5$. The distance to $-1$, as the Nyqist curve intersects the negative real axis, must hence be at least 0.5. This implies that the gain margin is at least 2.

# Solutions to Chapter 5. Stability

**5.1 a.** To be asymptotically stable, all eigenvalues of the system matrix $A$ must lie strictly within the left half plane (LHP). I.e. $\text{Re}(\lambda_i) < 0 \ \forall \ i$.

The eigenvalues of $A$ are given by the characteristic equation

$$\det(\lambda I - A) = 0$$

which in this case has two solutions, $\lambda_1 = -i$ and $\lambda_2 = i$. Since the eigenvalues do not lie strictly within the LHP, the system is not asymptotically stable.

**b.** If all the eigenvalues of $A$ lie strictly within the LHP, we are guaranteed stability. If any eigenvalue lies strictly in the RHP we have an unstable system. If, on the other hand, there are eigenvalues on the imaginary axis, the system can be either stable or unstable.

In this example there are no eigenvalues in the RHP. Additionally, all eigenvalues on the imaginary axis are distinct. This tells us that the system is stable.

**5.2** The closed loop transfer function is given by

$$G(s) = \frac{G_o}{1 + G_o} = \frac{K}{s^2 + 2s + K}$$

The poles of the closed loop system are given by the characteristic equation

$$s^2 + 2s + K = 0 \quad \Rightarrow \quad s = -1 \pm \sqrt{1 - K}$$

For $K = 0$ the roots $s_{1,2} = 0, -2$, i.e. the poles of the open loop system, are obtained. The closed loop system $G(s)$ has a double pole in $s = -1$ for $K = 1$. And as $K \to \infty$ the roots become

$$s_{1,2} = -1 \pm i\infty$$

The root locus, i.e. the roots of the characteristic equation as $K$ varies, is shown in figure 5.1 .

**5.3** The open loop transfer function of the system is

$$G_o(s) = \frac{K(s+10)(s+11)}{s(s+1)(s+2)} = K\frac{Q(s)}{P(s)}$$

The closed loop system becomes

$$G(s) = \frac{G_o(s)}{1 + G_o(s)} = \frac{KQ(s)}{P(s) + KQ(s)}$$

The characteristic equation is thus

$$\begin{aligned}
&P(s) + KQ(s) = 0 \\
\Leftrightarrow &s(s+1)(s+2) + K(s+10)(s+11) = 0 \\
\Leftrightarrow &s^3 + (3+K)s^2 + (2+21K)s + 110K = 0
\end{aligned}$$

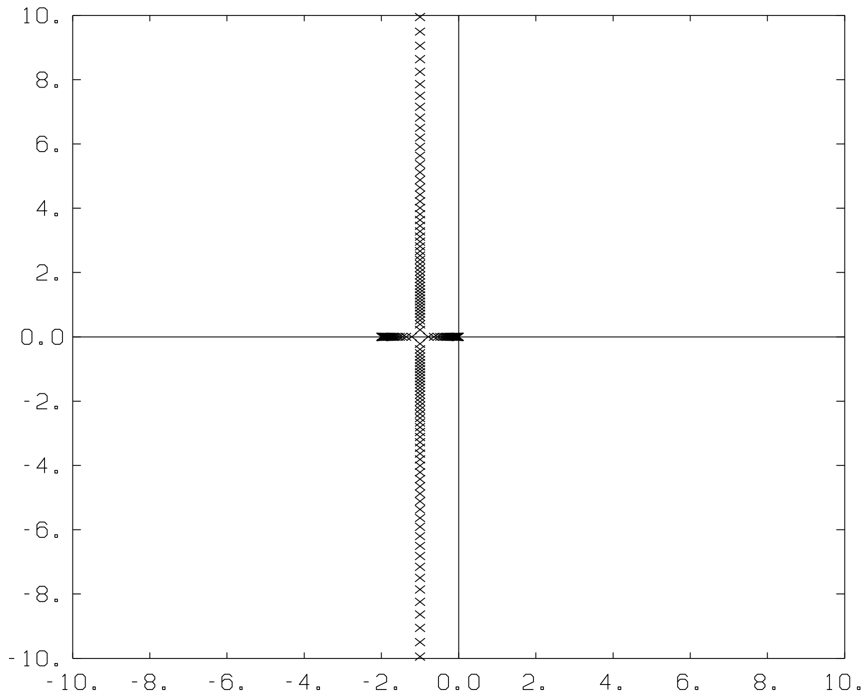



**Figure 5.1** Root locus of the system in assignment 5.2.

**a.** The criterion for stability is that all coefficients of the characteristic polynomial

$$s^3 + (3 + K)s^2 + (2 + 21K)s + 110K$$

are positive and that

$$(3 + K)(2 + 21K) > 110K$$

The inequality yields

$$K^2 - \frac{15}{7}K + \frac{2}{7} > 0$$

It is fulfilled for $K > 2$ and $K < 1/7$. The closed loop system is hence stable for

$$0 < K < \frac{1}{7}$$

and

$$K > 2$$

**b.** Find the root locus for the characteristic equation, $P(s) + KQ(s) = 0$

$$s(s + 1)(s + 2) + K(s + 10)(s + 11) = 0 \qquad (5.1)$$

Let $n$ = the degree of $P(s)$ and $m$ = the degree of $Q(s)$. The root locus has a maximum of $max(n, m) = 3$ branches.

Starting points:

$$P(s) = 0 \quad \Rightarrow \quad s = 0, -1, -2$$

Ending points:

$$Q(s) = 0 \quad \Rightarrow \quad s = -10, -11$$

The third branch will approach infinity.

To the right of each real point of the root locus, there must exists an odd number of zeros of $P(s)$ and $Q(s)$. The points $x$, which fulfill this are

$$x < -11 \quad -10 < x < -2 \quad -1 < x < 0$$

The root locus has $|n - m| = 1$ asymptote. This is the negative real axis, since the range $x < -11$ on the real axis belongs to the root locus.

The intersection with the imaginary axis is obtained by introducing $s = i\omega$ (4) above. This yields

$$-(3 + K)\omega^2 + 110K + i(-\omega^3 + (2 + 21K)\omega) = 0$$

The resulting equation has a solution $\omega = K = 0$ and

$$\begin{cases} -(3 + K)\omega^2 + 110K = 0 \\ \omega^2 - (2 + 21K) = 0 \end{cases}$$

gives $K = 1/7, \quad \omega = \pm\sqrt{5}$ or $K = 2, \quad \omega = \pm\sqrt{44}$.

We know from sub-assignment a that the closed loop system is unstable for $1/7 < K < 2$. Consequently, the root locus lies in the right half plane for these values of $K$. The principal shape of the root locus is shown in figure 5.2.

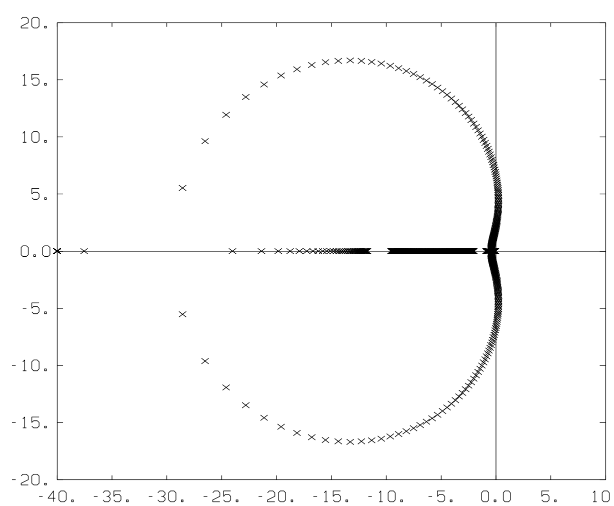



**Figure 5.2** Root locus of the system in assignment 5.3.

**5.4** The characteristic equation is

$$s^3 + 2s^2 + 3s + 7 = 0$$

The transfer function is stable if all coefficients are positive, which is the case, and if the product of the $s^2$- and $s^1$ coefficients is greater than the $s^0$ coefficient. The transfer function is therefore not stable, since $2 \cdot 3 < 7$.

**5.5 a.** The open loop transfer function of the system is

$$G_0(s) = \frac{K}{s(s+1)(s+2)}$$

The closed loop transfer function is thus

$$G_{cl}(s) = \frac{G_0(s)}{1+G_0(s)} = \frac{K}{s(s+1)(s+2)+K}$$

The system is asymptotically stable if all zeros of the characteristic polynomial

$$s(s+1)(s+2)+K = s^3+3s^2+2s+K$$

have negative real parts. This is the case if all coefficients are positive and if

$$3 \cdot 2 > K$$

The system is thus asymptotically stable if $0 < K < 6$.

**b.** Now we want to study the dependence of $K$ on the stationary error, as the reference increases as a linear function of time. The Laplace transform of the control error is given by

$$E(s) = \frac{1}{1+G_0}R(s) = \frac{s(s+1)(s+2)}{s(s+1)(s+2)+K}R(s)$$

With $r(t) = 0.1t$, i.e. $R(s) = 0.1/s^2$, we obtain

$$E(s) = \frac{0.1(s+1)(s+2)}{s(s(s+1)(s+2)+K)}$$

The signal $sE(s)$ has all poles in the left-half plane when $0 < K < 6$, according to sub-assignment a. For this case we can utilize the final value theorem

$$e(\infty) = \lim_{s\to 0} s\frac{0.1(s+1)(s+2)}{s(s(s+1)(s+2)+K)} = \frac{0.2}{K}$$

In order to obtain a stationary error less than 5 mV for the given reference, it is required that $K > 40$. For such large values of $K$ the system is, however, not stable. It it hence impossible to meet the specification.

**5.6** The closed loop transfer function is

$$G(s) = \frac{G_o(s)}{1+G_o(s)}$$

Introduce the notations $N$ = the number of RHP zeros of $1 + G_o(s)$, $P$ = the number of RHP poles of $1 + G_o(s)$ (= the number of RHP poles of $G_o(s)$). The open loop system $G_o(s)$ lacks RHP poles, which means that $P = 0$. We use Cauchy's argument principle to calculate $N$.

Let $K = 1$ and calculate $G_o(s)$ as $s$ traverses the curve $C$ according to figure 5.3. We note that $G_0$ has a pole in the origin. By drawing a small half circle around the origin in figure 5.3, we avoid singularities on the contour $C$. The curve $C$ enclosed the entire RHP as $R \to \infty$ and $r \to 0$.

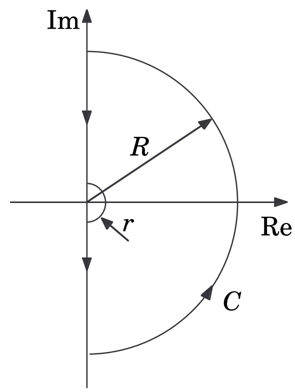



**Figure 5.3** The Nyquist contour $C$.

For the positive imaginary axis ($s = i\omega$) we have the following

$$
\begin{aligned}
G_o(i\omega) &= \frac{(1-i\omega)(2-i\omega)}{i\omega(1+i\omega)(2+i\omega)(1-i\omega)(2-i\omega)} \\
&= \frac{-3}{(\omega^2+1)(\omega^2+4)} + i\frac{(\omega^2-2)}{\omega(\omega^2+1)(\omega^2+4)}
\end{aligned}
$$

For large $\omega$ it holds that

$$
G_o(i\omega) \approx -\frac{3}{\omega^4} + \frac{i}{\omega^3}
$$

The Nyquist curve hence lies in the second quadrant and approaches the origin along the positive imaginary axis as $\omega \to \infty$.
For small $\omega$ it holds that

$$
G_o(i\omega) \approx -\frac{3}{4} - \frac{i}{2\omega}
$$

The Nyquist curve approaches infinity in the third quadrant, in parallel to the negative imaginary axis (Re $G_o(i\omega) = -\frac{3}{4}$) as $\omega \to 0$.
The intersection with the real axis is given by Im $G_o(i\omega) = 0$:

$$
\omega^2 - 2 = 0 \quad \Rightarrow \quad \omega = \sqrt{2}
$$

which yields the intersection point

$$G_o(i\sqrt{2}) = -\frac{1}{6}$$

For the negative imaginary axis ($s = -i\omega$) the following reasoning holds.

Since $G_o(s)$ has real coefficients, $G_o(-i\omega)$ is simply $G_o(i\omega)$ mirrored in the real axis, i.e. $G_o(-i\omega) = \mathrm{Re}\, G_o(i\omega) - i\mathrm{Im}\, G_o(i\omega)$.

Now we consider the large half circle ($s = Re^{i\varphi}$).

Because of

$$\lim_{R\to\infty} |G_o(Re^{i\varphi})| = 0$$

the large half circle is mapped onto the origin.

Finally we have the small half circle ($s = re^{i\varphi}$).

As $s$ traverses the small half circle, the argument $\varphi$ varies according to

$$\varphi = \frac{\pi}{2} \to 0 \to -\frac{\pi}{2}$$

For small $|s|$ it holds that

$$G_o(s) \approx \frac{1}{2s} = \frac{1}{2r}e^{-i\varphi}$$

The argument of $G_o(s)$ thus traverses the interval from $-\frac{\pi}{2}$ (through 0) to $\frac{\pi}{2}$ as the small half circle is traversed. Further, it holds that $|G_o(s)| \to \infty$ as $r \to 0$.

Figure 5.4 shows $G_o(s)$ as $s$ traverses the curve $C$. The solid-, dashed- and dash-dotted lines are the mappings of $s = i\omega$, $s = -i\omega$ and the small half circle, respectively. The large half circle is mapped onto the origin.

An enlargement of the interesting region around $-1$ is shown in figure 5.5. The intersection with the real axis is, as previously computed, $G_o(i\omega) = -1/6$.

As is evident from the figures, the point $-1$ is not encircled by $G_o(s)$ for $K = 1$, as $s$ traverses the curve $C$ in the direction of the arrow. Cauchy's argument principle gives

$$N - P = 0$$

Since $P = 0$ we have $N = 0$ and the closed loop system is hence stable for $K = 1$. This holds for all $K < 6$, since $N - P$ is constant for $K < 6$.

For $K > 6$, the point $-1$ will, however, be encircled twice in the positive direction by $G_o(s)$ and we obtain

$$N - P = 2 \quad \Rightarrow \quad N = 2$$

The closed loop system thus has two right half plane poles, which renders it unstable. The criterion for stability is hence $K < 6$.

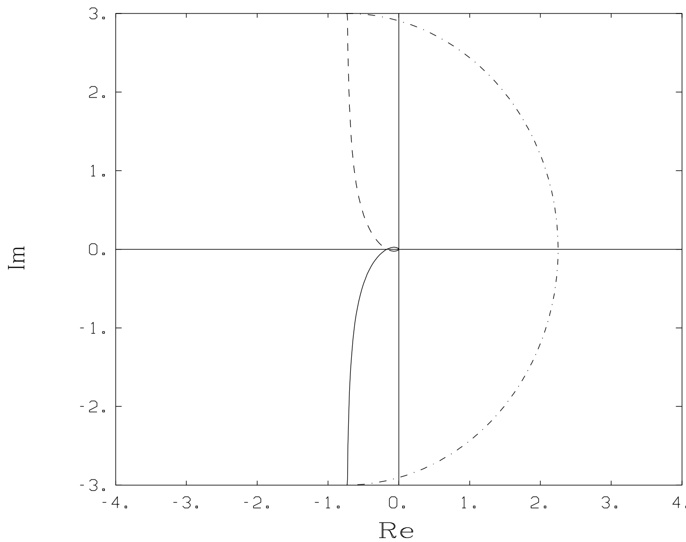



**Figure 5.4** $G_o(s)$ in assignment 5.6 as $s$ traverses the contour $C$ according to figure 5.4.

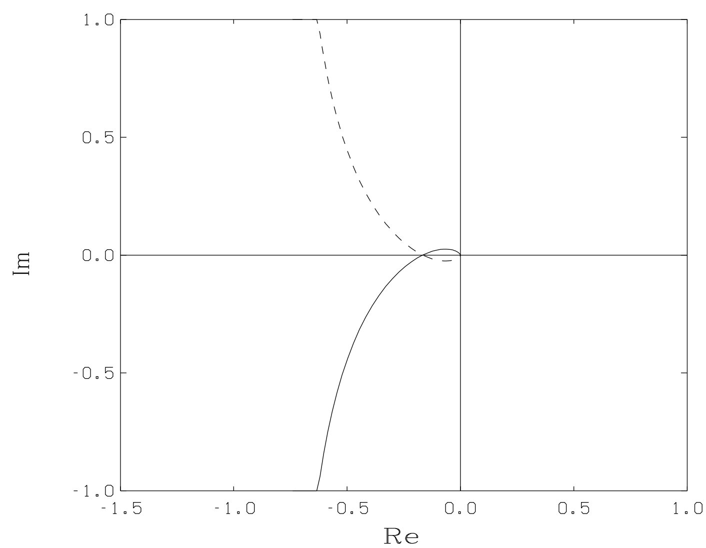



**Figure 5.5** $G_o(s)$ in assignment 5.6 as $s$ traverses the contour $C$. Enlargement of the region around $-1$.

*Comment.* $G_o(s)$ fulfills the requirements for the simplified Nyquist criterion, since the poles of the open loop system lie in the LHP, except for one pole in the origin. We can thus limit ourselves to study only the Nyquist curve $G_o(i\omega)$. We conclude that for $K < 6$, the point $-1$ lies to the left of the Nyquist curve, when it is traversed in the direction of increasing $\omega$. The closed loop system is thus stable. For $K > 6$, this criterion is not fulfilled and the closed loop system becomes unstable.

**5.7** According to the Nyquist theorem, the closed loop system is stable exactly for those $K > 0$, which are also

**a.** $K < 2$

**b.** $K < 1/1.5 = 2/3$

**c.** $K < 1/1.5 = 2/3$

**d.** $K < 1/(2/3) = 1.5$

**5.8** The Nyquist curve intersects the negative real axis when $\arg(G_P(i\omega)) = -\pi$, i.e. when

$$-3\arctan(\omega) = -\pi$$

This is fulfilled when

$$\omega = \tan\frac{\pi}{3} = \sqrt{3}$$

The intersection point is given by

$$|G_P(i\sqrt{3})| = \frac{1}{8}$$

This means that the system is stable for $K < 8$.

**5.9** The system is stable for

$$0 < K < \frac{1}{3.5} \quad \Leftrightarrow \quad 0 < K < 0.29$$

as well as

$$1 < K < \frac{1}{0.5} \quad \Leftrightarrow \quad 1 < K < 2$$

**5.10** The easiest way to solve the problem is through the Nyquist theorem. The transfer function of the process is

$$G_P(s) = \frac{e^{-9s}}{(1+20s)^2}$$

The phase shift of the process is

$$\arg G_P(i\omega) = -9\omega - 2\arctan(20\omega)$$

We seek the frequency for which the phase shift is $-180°$. It is obtained by solving the equation

$$-9\omega - 2\arctan(20\omega) = -\pi$$

The equation lacks an analytic solution. However, it can be solved numerically in several ways. The solution is

$$\omega_0 \approx 0.1$$

The next step is to determine the process gain for the given frequency.

$$|G(i\omega_0)| = \frac{1}{1+400\omega_0^2} = 0.2$$

This gives us the gain margin

$$A_m = \frac{1}{0.2} = 5$$

The gain $K = 5$ is thus the maximal admissible gain.

**5.11** The loop transfer function is

$$G_P(s)G_R(s) = e^{-sL} \cdot \frac{10(1+\frac{1}{2s})}{(1+10s)} = e^{-sL} \cdot \frac{5(1+2s)}{s(1+10s)}$$

The cross-over frequency is the frequency where the magnitude of the loop transfer function is equal to one.

$$|G_0(i\omega_c)| = \frac{5\sqrt{1+4\omega_c^2}}{\omega_c\sqrt{1+100\omega_c^2}} = 1$$

The equation can be solved numerically or analytically

$$\begin{aligned}
25(1+4\omega_c^2) &= \omega_c^2(1+100\omega_c^2) \\
\omega_c^4 - 0.99\omega_c^2 - 0.25 &= 0 \\
\omega_c^2 &\approx 1.199 \\
\omega_c &\approx 1.1 \quad \text{rad/min}
\end{aligned}$$

The phase at this frequency is

$$\arg G_0(i\omega_c) = \arctan 2\omega_c - \arctan 10\omega_c - 90^\circ - \omega_c L$$

The requirement of the phase margin, $\varphi_m \geq 10^\circ$, gives

$$\begin{aligned}
\varphi_m &= 180^\circ + \arg G_0(i\omega_c) \\
&= 180^\circ + \arctan 2\omega_c - \arctan 10\omega_c - 90^\circ - \omega_c L \\
&\approx 70^\circ - \omega_c L \geq 10^\circ
\end{aligned}$$

This gives the following limit for the time delay $L$:

$$L \leq \frac{60}{\omega_c} \cdot \frac{\pi}{180} = 1\,\text{min}$$

The time delay must therefore be less than one minute.

**5.12a.** True. $A_m = 1/|KG_P(i\omega_0)|$, where $\omega_0$ is the frequency for which the Nyquist curve intersects the negative real axis.

**b.** True. $\varphi_m = \pi + \arg G_P(i\omega)$ for $|G_P(i\omega)| = 1$.

**c.** False. As $K$ is decreased, all points on the Nyquist curve move closer to the origin. Thus the phase margin increases as $K$ is decreased.

**d.** True. The system is stable for $K = 1$ and all poles of $G_P(s)$ lie in the left half plane. Consequently, the simplified Nyquist criterion can be applied. For $K = 2$, the point $-1$ lies to the right of the Nyquist curve, when it is traversed as $\omega$ increases. The closed loop system is thus stable.

**5.13a.** This is the definition of the gain margin. From the plot one sees that the phase $-180°$ corresponds to the gain $\sim 0.4$. This yields the gain margin $1/0.4 = 2.5$.

**b.** This is the definition of the phase margin. From the plot one sees that for gain 1, the phase is approximately $-140°$. This yields a phase margin of approximately $180° - 140° = 40°$.

**5.14** The cross-over frequency and phase margin are read to be $\omega_c = 0.07$ and $\varphi_m = 40°$, respectively. The delay margin becomes

$$L_m = \frac{\varphi_m}{\omega_c} = \frac{40° \cdot \frac{\pi}{180°}}{0.07} = 10$$

# Solutions to Chapter 6. Controllability and Observability

**6.1 a.** From the controllability matrix

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} \beta & 1-\beta \\ 1 & -2 \end{pmatrix}$$

we obtain $\det W_s = -\beta - 1$, i.e. controllability for all $\beta \neq -1$.

**b.** The observability matrix

$$W_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & \gamma \\ 0 & -2\gamma \end{pmatrix}$$

has zero determinant independent of $\gamma$, i.e. the system is not observable for any value of $\gamma$.

**6.2** The controllability matrix is given by

$$W_s = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 4 & -8 & 16 \\ -2 & 4 & -8 \\ 1 & -2 & 4 \end{pmatrix}$$

The controllable states are described by the vector $(4, -2, 1)^T$.

**6.3**

$$W_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

We see that $W_o$ is singular ($\det W_o = 0$). The state $x$ is non-observable if and only if (iff)

$$W_o x = 0$$

We obtain a non-observable $x$ iff $x_1 + x_2 = 0$. The non-observable states are thus given by

$$x = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where $\alpha$ is a number $\neq 0$.

**6.4** The controllability matrix

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

is singular, yielding an uncontrollable system. We can, however, conduct a more detailed investigation. The system can be written

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 + u, & x_1(0) = 1 \\ \dfrac{dx_2}{dt} = -2x_2, & x_2(0) = 1 \end{cases}$$

Hence $x_2(t) = x_2(0)e^{-2t} = e^{-2t}$, independent of the applied control signal $u$. On the contrary, $x_1$ can be controlled by means of $u$, to take on any desired value. Thus, $x_2 \to 0$ as $t \to \infty$. The states $(x_1, x_2)$ which can be reached in finite time $t < \infty$ make up the band $0 < x_2 < 1$ in figure 6.1.

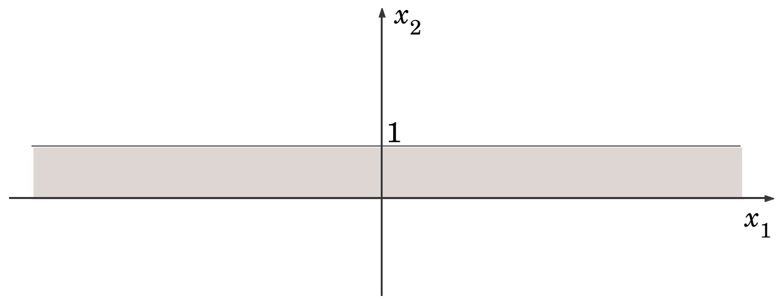



**Figure 6.1** Reachable states in assignment 6.4.

As a consequence, only the states $\begin{pmatrix} 3 \\ 0.5 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 0.1 \end{pmatrix}$ can be reached in finite time.

**6.5 a.** Controllable, since the controllablity matrix

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 8 & 4 & 2 \\ 2 & 6 & -7 & -16 \end{pmatrix}$$

has full rank (i.e. 2 in this case). This can be seen from e.g. the two first columns forming a non-singular $2 \times 2$ matrix.

**b.** For $u_2 = 0$ we obtain

$$B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The system is controllable, since

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & -7 \end{pmatrix}$$

has full rank.

**c.** For $u_1 = -2u_2$ we obtain

$$Bu = \begin{pmatrix} 1 & 8 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ -0.5 \end{pmatrix} u_1 = \begin{pmatrix} -3 \\ -1 \end{pmatrix} u_1 = B' u_1$$

The system is not controllable, since

$$W_s = \begin{pmatrix} B' & AB' \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix}$$

is not of full rank. (The column vectors are parallel.)

**6.6 a.**

$$W_s = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix}$$

We see that $W_s$ has rank 1, which is less than $n = 3$. The system is thus not controllable. The controllable states are spanned by the column vectors in $W_s$, i.e. $x = (0, 0, 1)^T$.

**b.**

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ -4 & -1 & 4 \end{pmatrix}$$

$W_o(s)$ has full rank, since $\det W_o = 11$. This means that the system is observable, i.e. there are no unobservable states.

**6.7 a.**

$$W_s = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 5 & -10 \\ 0 & 0 \end{pmatrix}$$

$W_s$ has rank 1, i.e. the system is not controllable. The states which can be reached in finite time from the origin are determined by the columns of $W_s$. The controllable states can be parametrized by $t$ as $x = t \cdot (1 \quad 0)^T$.

**b.** $G(s) = C(sI - A)^-1B + D = 5/(s + 2)$.

**c.** The following is a minimal state space representation of $G(s)$

$$\begin{cases} \dot{x} = -2x + 5u \\ y = x \end{cases}$$

# Solutions to Chapter 7. PID Control

**7.1 a.** The frequency function of the controller is given by

$$G_R(i\omega) = K\left(1 + i(\omega T_d - \frac{1}{\omega T_i})\right)$$

The gain and phase shift for a frequency $\omega$ are directly obtained from the gain function $A(\omega)$ and phase function $\phi\omega$ of the controller, respectively.

$$A(\omega) = |G_R(i\omega)| = K\sqrt{1 + (\omega T_d - \frac{1}{\omega T_i})^2}$$
$$\phi(\omega) = \arg G_R(i\omega) = \arctan(\omega T_d - \frac{1}{\omega T_i})$$

**b.** We immediately realize that the gain function $A(\omega)$ has a unique minimum for Im $A(i\omega) = 0$, which means that

$$\omega_{\min} = \frac{1}{\sqrt{T_i T_d}}$$

At this frequency the gain and phase shift are given by

$$A(\omega_{\min}) = K$$
$$\phi(\omega_{\min}) = 0$$

Note that the phase shift is negative for $\omega < \omega_{\min}$ (phase lag) and positive for $\omega > \omega_{\min}$ (phase lead).

**7.2 a.** The dashed gain curve is identical to the nominal one, except that it is raised by a factor 4. This is thus the case where $K$ has been multiplied by 4. Observe that the dashed phase curve is not visible in the plot since it coincides with the solid phase curve. The dotted gain curve differs from the nominal (solid) curve at low frequencies, for which it is lower. This indicates that $T_i$ has been increased, resulting in decreased low frequency gain. Also note that the phase curve has been raised for low frequencies. The last (dash-dotted) curve apparently corresponds to the case where $T_d$ has been increased. This is further indicated by the factor 4 raise of the gain curve for high frequencies. Also for this case, one can notice a certain increase in the phase, although for somewhat higher frequencies.

The dashed step response is faster and less damped than the nominal (solid) one. This is a characteristic sign of an increased gain $K$. The corresponding Bode plot confirmingly shows that the cut-off frequency, $\omega_c$, has increased (faster), while the phase margin has decreased (less damped. The dotted step response features a slow mode both in the reference- and load disturbance responses. Observe the relatively fast increase in the reference response to approximately 0.8, followed by a slow convergence to 1. This must be due to decreased integral action. The integral time $T_i$ has thus *increased* in this case. The corresponding Bode plot shows that $\omega_c$ is virtually unchanged.

This is seen in the step response by the fact that the first part has approximately the same speed as the nominal case, whereas the following slow settling is due to the decreased low frequency gain. The last (dash-dotted) step response obviously corresponds to an increase of the derivative time $T_d$. The reference response is subject to an fast initial increase, followed by a somewhat slower settling. This is seen in the Bode plot by the fact that the high frequency gain has increased, while the low frequency gain has remained unchanged. The load response is somewhat slower and more damped than in the nominal case.

**b.** The dashed gain curve is lowered by a factor 2 in comparison to the nominal one, corresponding to a decreased value of $K$. The dashed phase curve consequently coincides with the nominal (solid) case. The dotted gain curve has been increased for low frequencies, i.e. $T_i$ has *decreased*. The dash-dotted gain curve has been lowered at high frequencies, i.e. $T_d$ has decreased.

The dashed step response is slower and more damped than the nominal one. This indicates that $K$ has decreased, since neither a decrease in $T_i$ nor $T_d$ would yield a more damped step response. This is further confirmed by the Bode plot. The only case where $\omega_c$ has decreased is when $K$ has decreased. It is also the only case for which the phase margin has increased. The two remaining step responses are both less damped than the nominal one. In order to determine which of these corresponds to a decrease of $T_i$, we look at the corresponding Bode plot (the dotted one). This shows that the cutoff frequency $\omega_c$ has increased somewhat compared to the other nominal case. The dash-dotted Bode plot, however, shows that the decrease of $T_d$ has not changed $\omega_c$. The dotted step response is initially somewhat faster than the nominal (solid) one, whereas the dash-dotted one is initially approximately as fast as the nominal one. This implies that the dotted step response corresponds to a decrease in $T_i$, while the dash-dotted one corresponds to a decrease in $T_d$.

**7.3** The transfer function of the process is given by

$$G_P = \frac{C}{Js + D}$$

and the transfer function of the PI controller is given by

$$G_R = K\left(1 + \frac{1}{sT_i}\right)$$

We can now write down the closed loop transfer function $G_{cl}$ as

$$G_{cl} = \frac{G_R G_P}{1 + G_R G_P}$$

The characteristic polynomial is the denominator of $G_{cl}$

$$s^2 + \frac{D + CK}{J}s + \frac{CK}{JT_i}$$

and the desired characteristic polynomial is

$$s^2 + 2\zeta\omega s + \omega^2$$

Identification of coefficients yields

$$\begin{cases} K = \dfrac{2\zeta\omega J - D}{C} \\ T_i = \dfrac{2\zeta\omega J - D}{\omega^2 J} \end{cases}$$

**7.4** The transfer function of the process, $G_p$, is given by

$$\Theta = G_P I = \frac{k_i}{Js^2 + Ds} I$$

and the transfer function of the PID controller, $G_R$, is given by

$$I = G_R(\Theta_{ref} - \Theta) = K\left(1 + \frac{1}{sT_i} + sT_d\right)(\Theta_{ref} - \Theta)$$

where $\Theta_{ref}$ is the Laplace transform of the reference value of $\theta$. The transfer function of the closed loop system, $G$, is thus given by

$$\Theta = G\Theta_{ref} = \frac{G_R G_P}{1 + G_R G_P}\Theta_{ref}$$

The characteristic polynomial is given by the denominator of $G$ and is

$$s^3 + \frac{D + Kk_iT_d}{J}s^2 + \frac{Kk_i}{J}s + \frac{Kk_i}{JT_i}$$

We hence arrive at the polynomial equation

$$(s + \alpha)(s^2 + 2\zeta\omega s + \omega^2) = s^3 + (\alpha + 2\zeta\omega)s^2 + (2\alpha\zeta\omega + \omega^2)s + \alpha\omega^2$$

Identification of coefficients yields the equations

$$\begin{cases} \dfrac{D + Kk_iT_d}{J} = \alpha + 2\zeta\omega \\ \dfrac{Kk_i}{J} = 2\alpha\zeta\omega + \omega^2 \\ \dfrac{Kk_i}{JT_i} = \alpha\omega^2 \end{cases}$$

from which one can calculate the sought controller parameters

$$\begin{cases} K = \dfrac{J}{k_i}(2\alpha\zeta\omega + \omega^2) \\ T_i = \dfrac{2\zeta}{\omega} + \dfrac{1}{\alpha} \\ T_d = \dfrac{\alpha + 2\zeta\omega - D/J}{2\alpha\zeta\omega + \omega^2} \end{cases}$$

**7.5 a.** The transfer function of the controller is

$$G_r(s) = K(1 + \frac{1}{sT_i}) = (1 + \frac{1}{s}) = \frac{s+1}{s}$$

The low frequency asymptote becomes

$$G_r(s) \approx \frac{1}{s}$$

I.e. the gain curve is a straight line with slope = -1 and $\arg G_r(i\omega) = -90°$. The slope of the gain curve increases to 0 at the corner frequency $\omega_1 = 1$.

The high frequency asymptote is $G_r(s) \approx 1$ with $|G_r(i\omega)| = 1$, i.e. slope = 0 and $\arg G_r(i\omega) = 0$. The corresponding Bode plot is shown in figure 7.1.

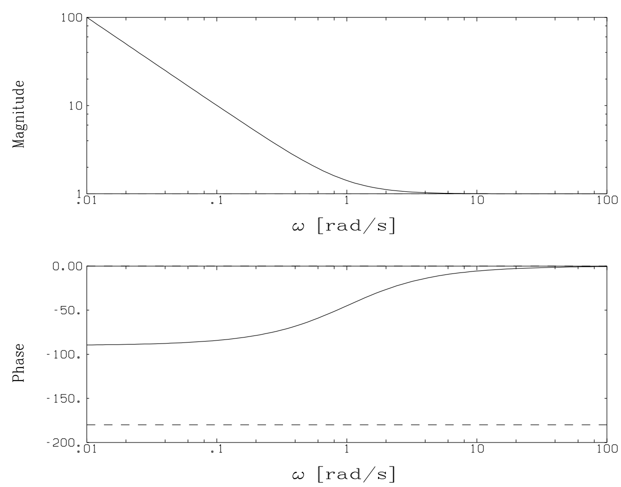



**Figure 7.1** Bode plot of a PI controller with $K = 1$ and $T_i = 1$.

**b.** The transfer function of the controller is

$$G_r(s) = K(1 + T_d s) = 1 + s$$

The low frequency asymptote becomes $G_r(s) \approx 1$, i.e. the gain curve is a straight line with magnitude 1 and slope = 0 and the phase curve is described by $\arg G_r(i\omega) = 0°$. The slope of the gain curve increased to +1 at the corner frequency $\omega_1 = 1$.

The high frequency asymptote is given by $G_r(s) \approx s$, i.e. the slope of the gain curve is +1 and the phase curve is described by $\arg G_r(i\omega) = +90°$. The corresponding Bode plot is shown in figure 7.2.

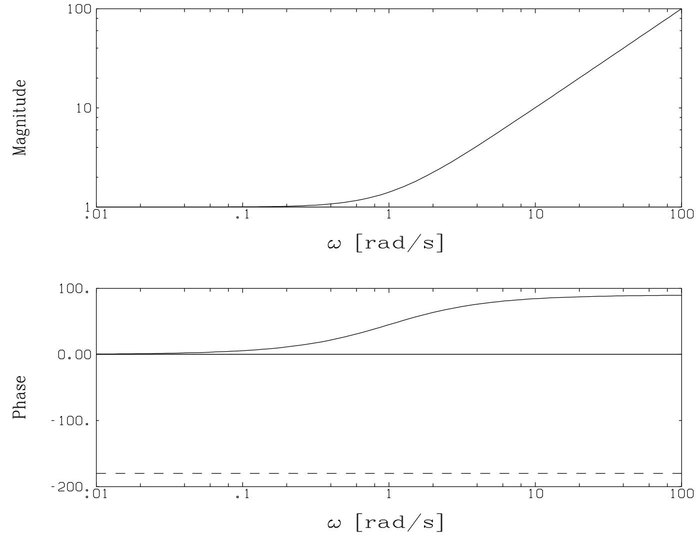



**Figure 7.2** Bode plot of a PD controller with $K = 1$ and $T_d = 1$.

**7.6** The transfer function of the process can be factored as

$$G_P(s) = \frac{e^{-9s}}{(1+20s)^2} = G_1(s) \cdot G_2(s)$$

where

$$G_1(s) = \frac{1}{(1+20s)^2}$$

and

$$G_2(s) = e^{-9s}$$

First draw the Bode plot of $G_1(s)$. This system has LF gain 1 and corner frequency $\omega_1 = 1/20 = 0.05$ rad/s. the slope of the HF asymptote is $-2$.

Then compute the Bode plot of $G_P(s)$ by superpositioning the phase curve of $G_1(s)$ with the contribution from the delay $G_2(s)$, which has gain $= 1$ and phase shift

$$\varphi = \arg G_2(i\omega) = -9\omega$$

The result is shown in figure 7.3.

The bode plot in figure 7.3 tells that the phase shift of the process is $-180°$ for $\omega = \omega_o = 0.1$ rad/s. The corresponding gain is $|G(i\omega_P)| = 0.2$. The critical gain $K_c$ thus becomes

$$K_c = \frac{1}{0.2} = 5$$

and the period is

$$T_o = \frac{2\pi}{\omega_o} = 63$$

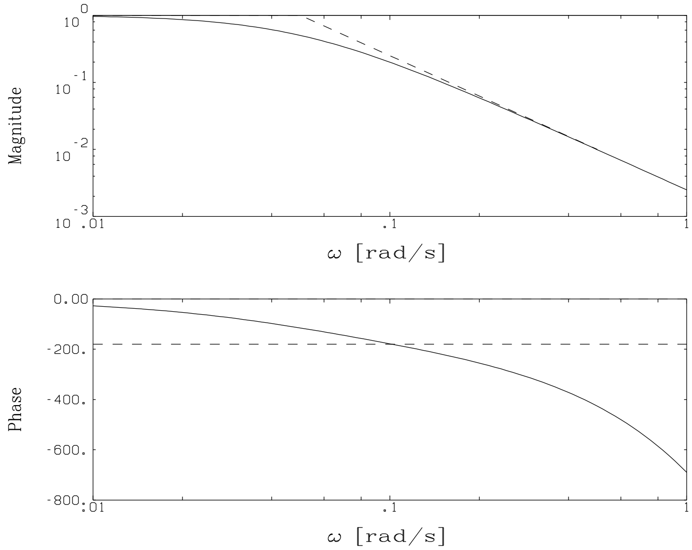



**Figure 7.3** Bode plot of the cement oven in assignment 7.6.

The obtained controller parameters are thus

$$\begin{cases} K = 0.45K_c = 2.25 \\ T_i = T_o/1.2 = 53 \end{cases}$$

**7.7** Step response method: With the customary notion we obtain $a = 0.3$ and $b = 0.8$. The controller parameters become $K = 1.2/a = 4$, $T_i = 2b = 1.6$ and $T_d = b/2 = 0.4$.

Frequency method: The Nyquist curve intersects the negative real axis in $-1/3$ for $\omega = 1$ rad/s, which yields $T_o = 2\pi/\omega = 2\pi$ and $K_c = 3$. The controller parameters become $K = 0.6K_c = 1.8$, $T_i = T_o/2 = 3.1$ and $T_d = T_o/8 = 0.8$.

**7.8 a.** The step response of the system is shown in figure 7.4.

From the figure we obtain (with the customary notion) $a = b = 1$. This yields the controller parameters $K = 1.2/a = 1.2$, $T_i = 2b = 2$ and $T_d = b/2 = 0.5$.

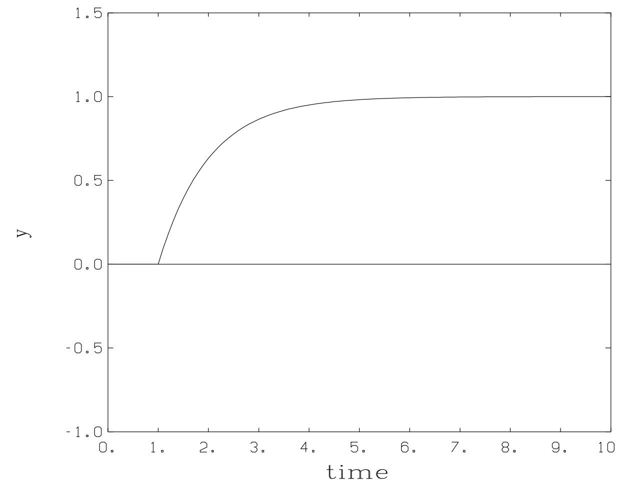



**Figure 7.4** Step response of $G(s) = e^{-s}/(s+1)$.

**b.** The resonance frequency is determined by $\arg G(i\omega) = -\arctan\omega - \omega = -\pi$. Numerical solution yields $\omega \approx 2.03$, resulting in $T = 2\pi/\omega = 3.1$.

Further, $K_c = 1/|G(i\omega_c)| = 2.26$, which gives $K = 1.4$, $T_i = 1.5$ and $T_d = 0.39$.

**7.9** The Bode plot of the system is shown in figure 7.5.

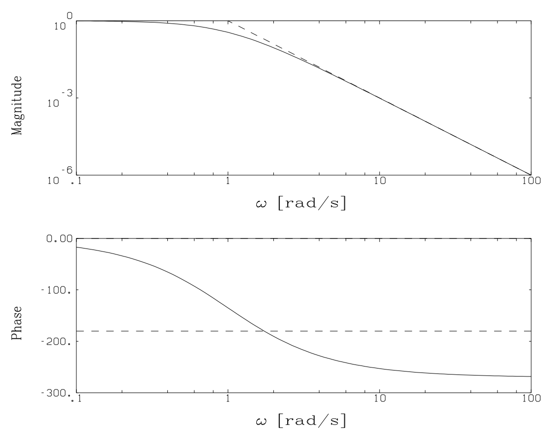



**Figure 7.5** Bode plot of $G(s) = 1/(s+1)^3$.

The phase curve passes $-180°$ when

$$\arg G(i\omega_c) = -3\arctan(\omega_c) = -\pi \quad \Rightarrow$$

$$\omega_c = \tan(\frac{\pi}{3}) = \sqrt{3} = 1.732$$

At this frequency, the gain is

$$|G(i\omega_c)| = (1+\omega_c^2)^{-3/2} = 1/8 = 0.125$$

This yields $K_c = 8$ and the period $T_o = 2\pi/\omega_c = 3.6$.

The controller parameters become $K = 0.6K_c = 4.8$, $T_i = T_o/2 = 1.8$ and $T_d = T_o/8 = 0.45$.

**7.10a.** The figure does not allow for any greater precision. Draw the tangent of the step response where the derivative attains a maximum and study the intersection of the tangent and the two coordinate axis. The parameter $a$ is given by the distance between 0 and the intersection with the vertical axis, whereas $b$ is given by the distance between 0 and the intersection with the horizontal axis. In our example we have $a = 0.65$ and $b = 4$. From the table we obtain the following controller parameters: $K = 1.9$, $T_i = 8$ and $T_d = 2$.

**b.** The critical gain $K_c$ is the gain which causes the Nyquist curve to pass through -1. In our case we have $K_c = 1/0.55 = 1.8$. The critical period $T_0$ corresponds to the frequency at 'o', i.e. $T_0 = 2\pi/\omega = 14.6$. This yields the controller parameters: $K = 1.1$, $T_i = 7.3$ and $T_d = 1.8$.

**c.** The value of $K$ obtained from the last method is smaller than the values obtained through Ziegler-Nichol's methods.

# Solutions to Chapter 8. Lead-Lag Compensation

**8.1 a.** In the frequency domain the bandwidth gives a measure of the system's speed. A high bandwidth, $\omega_b$, yields sinusoids with angular frequencies lower than $\omega_b$ and an amplification of at least $1/\sqrt{2}$. However, frequencies above $\omega_b$ will be damped.

In the singularity chart, the distance of the dominant poles to the origin, are the main measure of the system's speed.

Answer: B,D.

**b.** In the frequency domain, the resonance peak mainly gives information concerning the overshooting behavior of the system.

A way to motivate this is to draw a Bode plot of the closed loop system.

$$G(s) = \frac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2}$$

with varying $\zeta$. Small $\zeta$ yield a high resonance peak, which decreases with increasing $\zeta$.

In the singularity chart the angle $\varphi$ is a measure of the system's overshooting behavior (the relative damping $\zeta = \cos\varphi$).

Answer: C,F.

**8.2** Generally it is required that $|G_2(i\omega_c)| > 1$ in order for $\omega_c$ to increase.

**A** The speed of the system increases, but simultaneously its stability is reduced since the phase margin decreases.

**B** $|G_2| < 1$ for all $\omega$, resulting in decreased cross-over frequency and speed.

**C** Cf. B.

**D** $|G_2| = 1$ for all $\omega$, leaving the cross-over frequency unaffected.

**8.3** The process is connected in a feedback loop with a proportional controller. By adding a compensation link one wants to decrease the ramp error of the compensated system by a factor 10. Simultaneously, a small decrease in stability (phase margin) is accepted, resulting in a certain decrease of the system's transient behavior.

We can affect the ramp error by introducing a phase lag compensation link

$$G_k(s) = M\frac{s+a}{sM+a}$$

The resulting ramp error becomes

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+G_k(s)G_P(s)} \cdot \frac{1}{s^2} \\ &= \lim_{s\to 0} \frac{(sM+a)(s+1)(s+2)}{s(sM+a)(s+1)(s+2) + KM(s+a)} = \frac{2}{KM}\end{aligned}$$

By choosing $M = 10$ ($K = 1$) the ramp error is reduced to 0.2.

Now it remains to decide a value for $a$. The phase lag link contributes to a phase lag in the open loop. The phase lag is largest around the frequency $\omega = a/\sqrt{M}$. In order not to compound the transient behavior of the closed loop system excessively, $a$ must be chosen such that the phase around the cross-over frequency is left unaffected. This can be achieved by choosing $a$ adequately small. However, a overly small value of $a$ results in a long time before the ramp error decreases to 0.2. Let $\omega_c$ denote the cross-over frequency of the uncompensated system. At this frequency, the compensation link has a phase contribution

$$\arg G_k(i\omega_c) = \arctan \frac{\omega_c}{a} - \arctan \frac{M\omega_c}{a}$$

A simple rule of thumb is to choose $a = 0.1\omega_c$. In our example it means that the compensation link contributes with a phase shift of

$$\arg G_k(i\omega_c) = \arctan 10 - \arctan 100 \approx -5.1°$$

From the Bode plot of the uncompensated system (see figure 8.1) we read the cross-over frequency $\omega_c$ to be 0.4 rad/s. The compensation link thus becomes

$$G_k(s) = 10\frac{s + 0.04}{10s + 0.04} = \frac{s + 0.04}{s + 0.004}$$

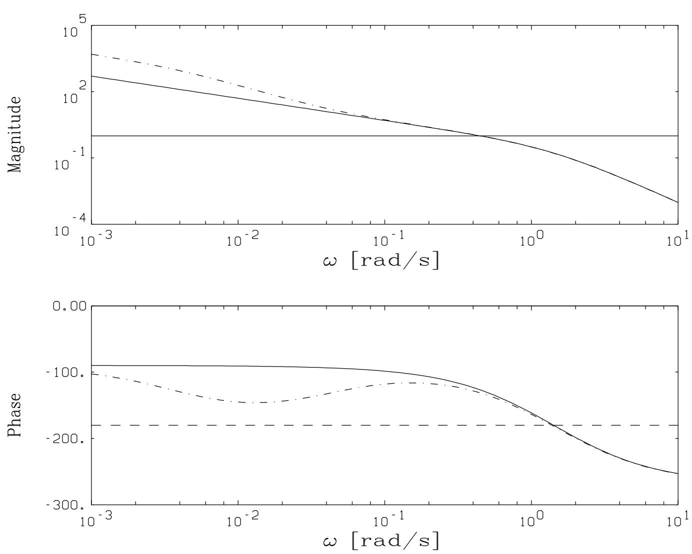



**Figure 8.1** Bode plot of the uncompensated open loop system (solid line) and compensated open loop system (dash-dotted line) in assignment 8.3. $K = 1$ for both cases.

In Figure 8.1 Bode plots for both the uncompensated open loop system $KG_P(s)$ and the compensated open loop system $KG_k(s)G_P(s)$ are shown.

The compensation link alters the transient behavior of the system. Figure 8.2 shows how the overshoot of the step response has increased, compared to the uncompensated system. Also the settling time has increased, partly due to the slow mode in the compensation link.

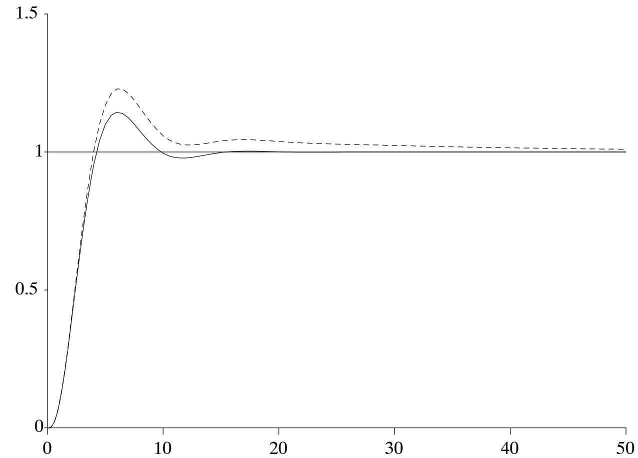



**Figure 8.2** Step responses of the uncompensated closed loop system (solid line) and the compensated closed loop system (dashed line) in assignment 8.3.

The purpose of introducing the compensation link was to decrease the ramp error. Figure 8.3 shows the error $e = r - y$ for both the uncompensated and compensated systems, with $r = t$.

As seen from the figure, the compensated system fulfills the criterion of a ramp error less than 0.2.

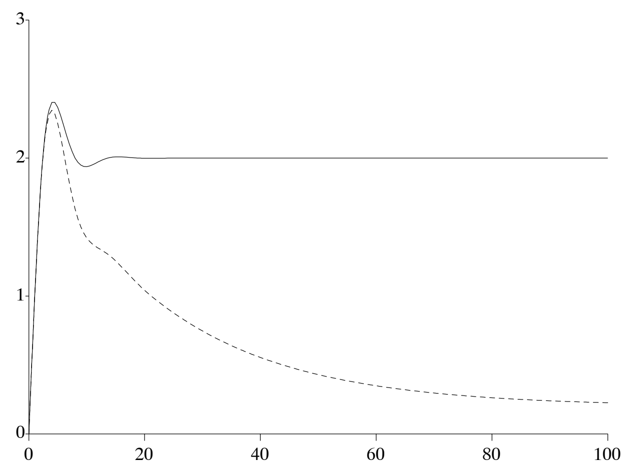



**Figure 8.3** Ramp error of the uncompensated system (solid line) as well as the compensated system (dashed line) in assignment 8.3.

**8.4** Use a phase lag compensation link

$$G_k(s) = KN\frac{s+b}{s+bN}$$

The cross-over frequency $\omega_c$ of the uncompensated system can be read from the Bode plot in figure 8.4. One can also determine $\omega_c$

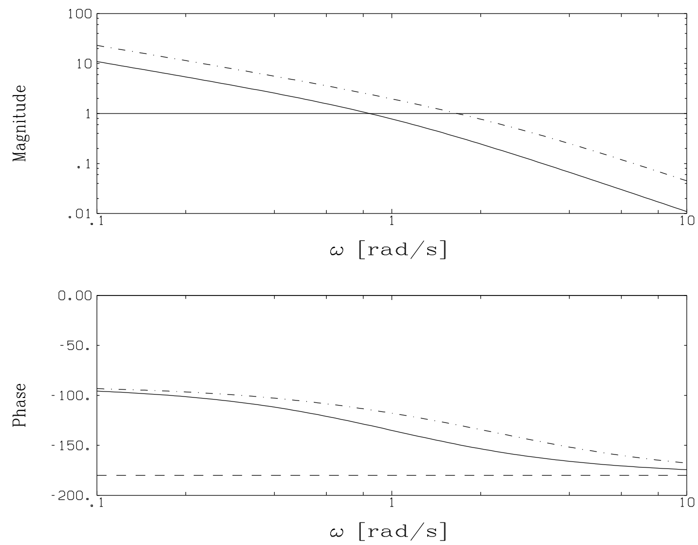



**Figure 8.4** Bode plot of the uncompensated open loop system (solid line) as well as the compensated open loop system (dash-dotted line) in assignment 8.4.

from the equation

$$|G_P(i\omega_c)| = \frac{1.1}{\omega_c\sqrt{\omega_c^2+1}} = 1$$

This yields $\omega_c = 0.84$. The new cross-over frequency is chosen to be $\omega_c^* = 1.68$. The phase shift of the uncompensated system at the frequency $\omega_c$ is

$$\arg G_P(i\omega_c) = -90^o - \arctan(0.84) = -130^o$$

In order not to decrease the phase margin, it must hold that

$$\arg(G_k(i\omega_c^*)G_P(i\omega_c^*)) \geq \arg G_P(i\omega_c)$$

We have

$$\arg G_P(i\omega_c^*) = -90^o - \arctan(1.68) = -149^o$$

For the compensation link it must hence hold that

$$\arg G_k(i\omega_c^*) \geq 19^o$$

From the collection of formulae we find that $N = 2$ is adequate. The compensation link has its maximal phase shift at the frequency $b\sqrt{N}$. This shall occur at the new cross-over frequency, i.e.

$$\omega_c^* = b\sqrt{N} \quad \Rightarrow \quad b = \frac{\omega_c^*}{\sqrt{N}} = 1.2$$

Now choose $K$ such that $\omega_c^*$ becomes the actual cross-over frequency (observe that $|G_k(i\omega_c^*)| = K\sqrt{N}$)

$$|G_k(i\omega_c^*)G_P(i\omega_c^*)| = 1 \quad \Rightarrow \quad K = 2.1$$

We thus obtain the compensation link

$$G_k(s) = 4.2\frac{s+1.2}{s+2.4}$$

Figure 8.4 shows the Bode plot of the uncompensated open loop system $G_P(s)$ as well as the compensated open loop system $G_k(s)G_P(s)$. Figure 8.5 shows the step responses of the uncompensated and compensated systems.

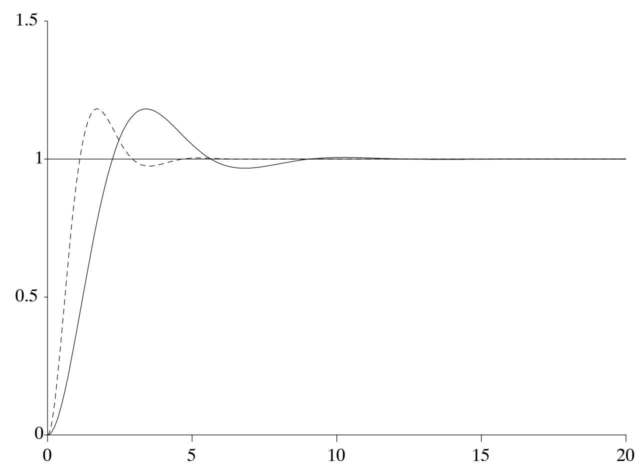



**Figure 8.5** Step response of the uncompensated closed loop system (solid line) as well as the compensated closed loop system (dashed line) in assignment 8.4.

**8.5** We choose a phase lead link

$$G_k(s) = K_K \cdot N\frac{s+b}{s+bN}$$

The specification implies that the low frequency gain shall not decrease (which would increase the stationary error). The cross-over frequency shall increase by a factor 3 and the phase margin shall remain unchanged.

The open loop transfer function is

$$G_0(s) = G_k(s)G_1(s) = K_K \cdot N\frac{s+b}{s+bN} \cdot \frac{1}{s(s+1)(s+2)}$$

The Bode plot of $G_1$ is presented in figure 8.6. From this, or from numerical calculations, the cross-over frequency is determined to $\omega_c = 0.45$ rad/s and the phase margin is $\varphi_m = 53°$. The new cross-

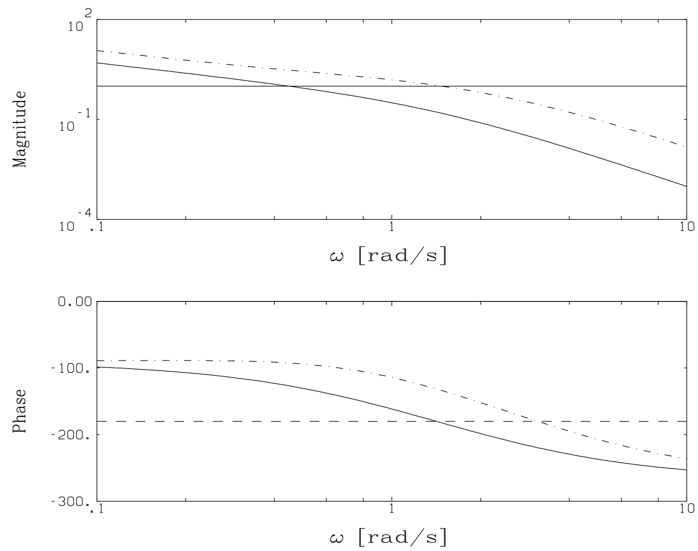



**Figure 8.6** Bode plot of the uncompensated system $G_1$ (solid line) and compensated system $G_k G_1$ (dash-dotted line) in assignment 8.5.

over frequency shall thus be $\omega_c^* = 3 \cdot \omega_c = 1.35$ rad/s with unchanged phase margin. Since $\arg G_1(i\omega_c^*) \approx -180°$, the phase curve must be raised approximately 50° by $G_k$.

From the collection of formulae it is obtained that $N = 8$ gives a maximal phase lead of approximately 50°. The phase lead is maximal at the frequency $b\sqrt{N} = \omega_c^*$, yielding $b = 0.48$. The gain shall be unity at the new cross-over frequency $\omega_c^*$, i.e.

$$|G_k(i\omega_c^*)| \cdot |G_1(i\omega_c^*)| = 1$$

The magnitude of the compensator is $|G_k(i\omega_c^*)| = K_K\sqrt{N}$. Numerical calculations give $|G_1(i\omega_c^*)| = 0.18$. Hence

$$K_K = \frac{1}{\sqrt{N} \cdot 0.18} = 2.0$$

The step response of the uncompensated and compensated systems, respectively, are shown in figure 8.7 and the ramp response is shown in figure 8.8. Since $K_K > 1$ the stationary error become smaller than before, thus fulfilling the specifications.

**8.6** From the Bode plot of $G_o(s)$ (see figure 8.9) we obtain $\phi_m = 20°$ and $\omega_c = 0.7$ rad/s. Unchanged speed necessitates a compensation link which does not affect the cross-over frequency. We hence need a phase lead of $\Delta\phi_m = 30°$ at $\omega = \omega_c = 0.7$ rad/s. We utilize a phase lead compensation link

$$G_k(s) = KN\frac{s+b}{s+bN}$$

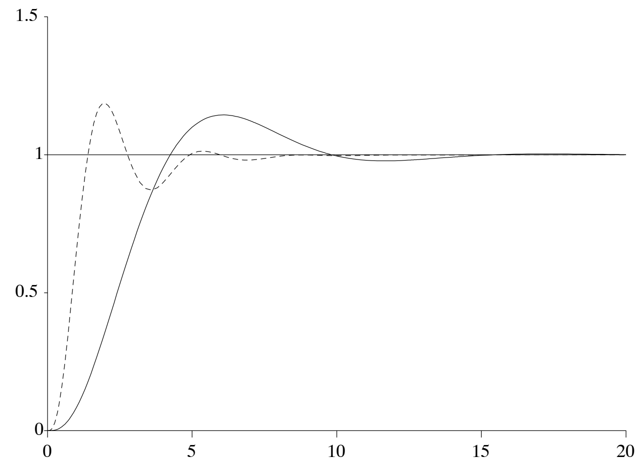



**Figure 8.7** The step response of the uncompensated closed loop system (solid line) as well as the compensated closed loop system (dashed line) in assignment 8.5.

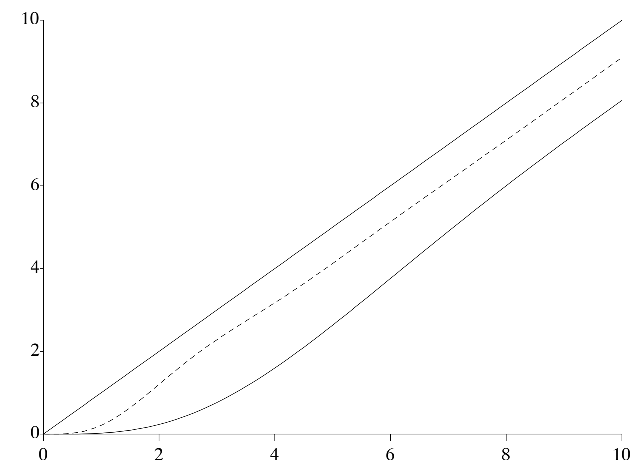



**Figure 8.8** The ramp response of the uncompensated system (solid line) as well as the compensated system (dashed line) in assignment 8.5.

**1.** The sample curves in the collection of formulae yield $N = 3$.

**2.** $b\sqrt{N} = \omega_c \quad \Rightarrow \quad b = \frac{0.7}{\sqrt{3}} = 0.40$

**3.** $|G_k(i\omega_c)G_o(i\omega_c)| = K\sqrt{N} \cdot 1$ gives $K = \dfrac{1}{\sqrt{N}} = 0.58$

The compensation link thus becomes

$$G_k(s) = 0.58 \cdot 3\frac{s+0.4}{s+1.2}$$

Finally, we calculate the resulting stationary error

$$\begin{aligned} E(s) &= \frac{1}{1+G_kG_o}R(s) \\ &= \frac{s(s+0.5)(s+3)(s+bN)}{s(s+0.5)(s+3)(s+bN)+2KN(s+b)}R(s) \end{aligned}$$

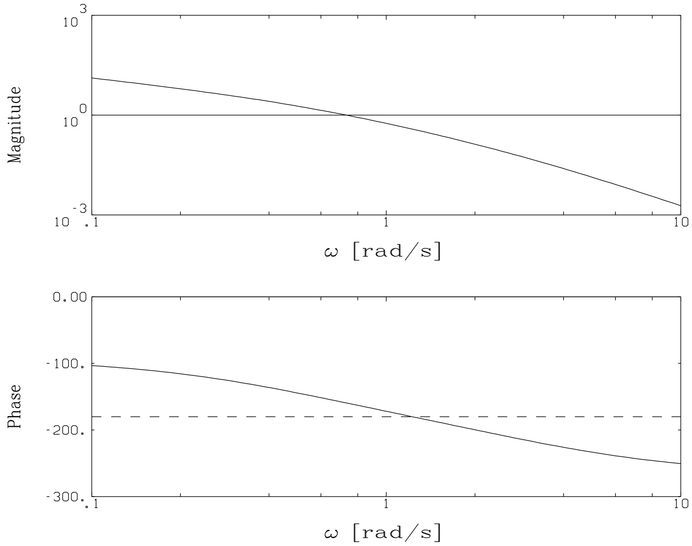



**Figure 8.9** Bode plot of $G_o(s)$ in assignment 8.6.

With $R(s) = 1/s^2$ the stationary ramp error becomes

$$\lim_{s\to 0} sE(s) = \frac{1.5}{2K} = 1.29$$

which fulfills the specification. Figure 8.10 shows the step response of the system before and after the compensation. The ramp error is shown in figure 8.11. The fact that the ramp error is increased by the compensation is due to $K < 1$.

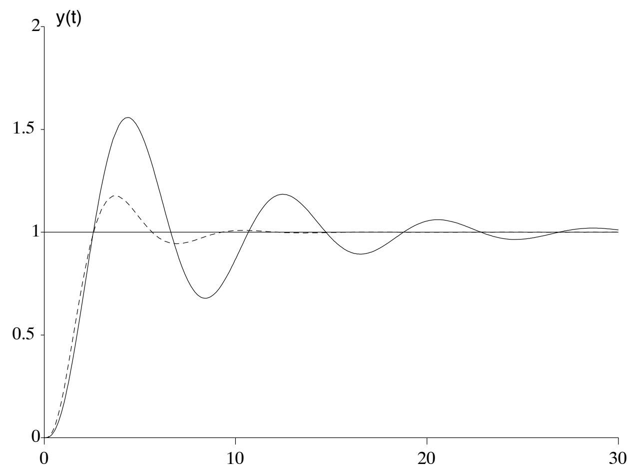



**Figure 8.10** Step response of the uncompensated closed loop system (solid line) and compensated system (dashed line) in assignment 8.6.

**8.7** We know that a phase lag compensation link dimensioned according to the rules of thumb will decrease the phase margin by approximately 6°, which yields a certain decrease of stability. In order not

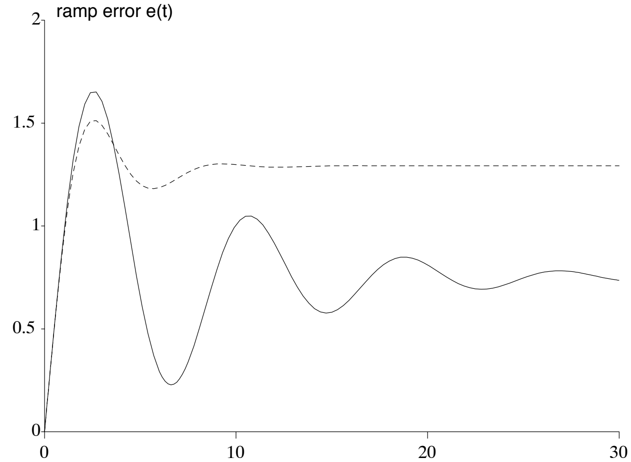



**Figure 8.11** Ramp error of the uncompensated closed loop system (solid line) as well as the compensated closed loop system (dashed line) in assignment 8.6.

to obtain an excessive overshoot, we start by decreasing the gain of the process in order to increase the phase margin.

From the Bode plot of the process (see figure 8.12) we find that $G-1$

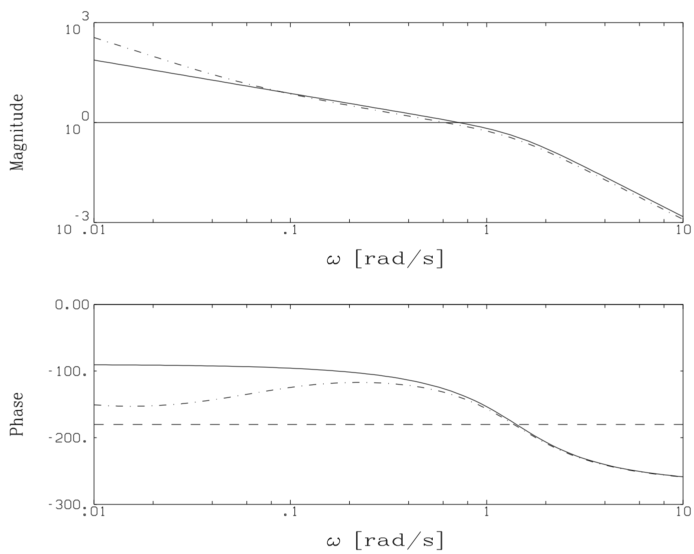



**Figure 8.12** Bode plot of the uncompensated open loop system (solid line) as well as the compensated open loop system (dash-dotted line) in assignment 8.7.

has a phase shift of $-133°$ at the cross-over frequency $\omega_c = 0.7$. At the frequency $\omega_c^* = 0.6$ we have the phase shift $-133° + 6° = -127°$ and the gain $|G_1(\omega_c^*)| = 1.2$.

By decreasing the open loop gain by a factor 1.2 we obtain the new cross-over frequency $\omega_c^*$ and a phase margin increase of 6°. Since we cannot affect the process gain directly, we equivalently let $K = 1/1.2 = 0.83$ in the compensation link.

The main problem is to decrease the stationary ramp error to $e_1 \leq$

0.1. The final value theorem gives

$$
\begin{aligned}
e(\infty) &= \lim_{s\to 0} sU(s)\frac{1}{1+G_k(s)G_1(s)} \\
&= \lim_{s\to 0} s\frac{1}{s^2}\frac{(s+a/M)s(s^2+2s+2)}{(s+a/M)s(s^2+2s+2)+1.5K(s+a)} \\
&= \frac{2}{1.5KM} \le 0.1
\end{aligned}
$$

which yields $M \ge 16$. Choose $M = 16$. According to the rule of thumb we let $a = 0.1\omega_c^* = 0.06$. The chosen compensation link thus becomes

$$
G_k(s) = 0.83\frac{s+0.06}{s+0.00375}
$$

Figure 8.13 shows the step response before and after the compensation. The ramp errors of the uncompensated closed loop system and

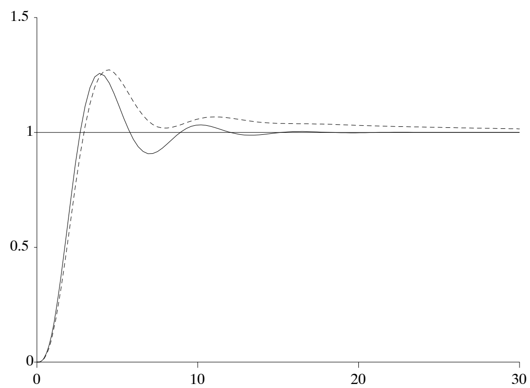



**Figure 8.13** Step response of the uncompensated closed loop system (solid line) as well as the compensated closed loop system (dashed line) in assignment 8.7.

the compensated closed loop system are shown in figure 8.14.

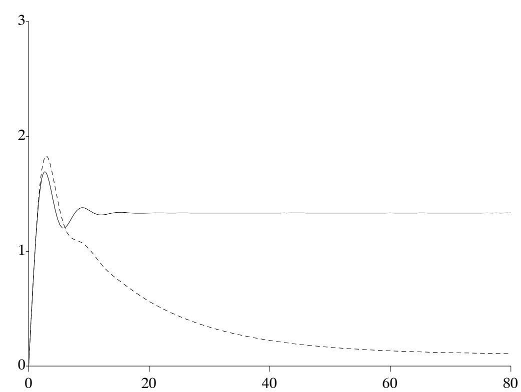



**Figure 8.14** Ramp error of the uncompensated closed loop system (solid line) as well as the compensated closed loop system (dashed line) in assignment 8.7.

*Comment.* Since we have decreased the open loop gain we obtain a decreased cross-over frequency and hence a somewhat slower system. In figure 8.13 one especially notes the slow mode which appears as the process settles. It is caused by the slow pole of the controller in combination with the low gain. The rise time and damping are, however, virtually unaffected. An alternative to decreasing the open loop gain, in order to maintain the desired phase margin, would be to introduce a phase lead compensation link.

# Solutions to Chapter 9. State Feedback and Kalman Filtering

**9.1 a.** True. Since the system is controllable, one can place the poles of the closed loop system arbitrarily by means of linear feedback from all state variables.

**b.** False. A linear state feedback does not affect the zeros of the closed loop system.

**c.** True if the system is observable.

**d.** True if the system is observable.

**9.2** The closed loop system becomes

$$\begin{cases} \dot{x} = (A - BL)x + Bl_r r \\ y = Cx \end{cases}$$

The characteristic equation is thus

$$\det(sI - A + BL) = s^2 + (3 + l_1 + 2l_2)s + 2(1 + l_1 + l_2) = 0$$

We need $(s+4)^2 = s^2 + 8s + 16 = 0$. Identification of coefficients yields $l_1 = 9, \quad l_2 = -2$. The closed loop transfer function is $G(s) = C(sI - A + BL)^{-1}Bl_r$. The stationary gain is $G(0)$ is unity if

$$G(0) = C(-A + BL)^{-1}Bl_r = \frac{l_r}{4} = 1$$

yielding $l_r = 4$.

**9.3 a.** The characteristic polynomial of the closed loop system is given by

$$\det(sI - (A - BL)) = \begin{vmatrix} s + 0.5 + 3l_1 & 3l_2 \\ -1 & s \end{vmatrix} = s^2 + (0.5 + 3l_1)s + 3l_2$$

The desired characteristic polynomial is

$$(s + 4 + 4i)(s + 4 - 4i) = s^2 + 8s + 32$$

Identification of coefficients yields

$$L = \begin{pmatrix} 5/2 & 32/3 \end{pmatrix} = \begin{pmatrix} 2.5 & 10.7 \end{pmatrix}$$

The closed loop system transfer function is $G_{yr}(s) = C(sI - A + BL)^{-1}Bl_r$. The stationary gain is unity if

$$G_{yr}(0) = C(-A + BL)^{-1}Bl_r = \frac{3}{32}l_r = 1$$

which yields $l_r = 32/3$.

**b.** According to a rule of thumb, the observer poles shall be chosen 1.5–2 times faster than the state feedback. Place the poles of the Kalman filter in e.g. −8, leading to the following characteristic polynomial

$$(s+8)(s+8) = s^2 + 16s + 64$$

The characteristic polynomial of the Kalman filter is given by

$$\det(sI-(A-KC)) = \begin{vmatrix} s+0.5 & k_1 \\ -1 & s+k_2 \end{vmatrix} = s^2+(0.5+k_2)s+0.5k_2+k_1$$

Identification of coefficients yields

$$K = \begin{pmatrix} 225/4 \\ 31/2 \end{pmatrix} = \begin{pmatrix} 56.25 \\ 15.5 \end{pmatrix}$$

**9.4** Introduce the state vector $x = \begin{pmatrix} \dot{\theta} & \theta & \dot{z} \end{pmatrix}^T$. The state space description of the craft dynamics is

$$\dot{x} = \begin{pmatrix} \ddot{\theta} \\ \dot{\theta} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} K_1 u \\ \dot{\theta} \\ K_2\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & K_2 & 0 \end{pmatrix} x + \begin{pmatrix} K_1 \\ 0 \\ 0 \end{pmatrix} u = Ax + Bu$$

The control law is given by

$$u = u_{\text{ref}} - l_1\dot{\theta} - l_2\ddot{z} - l_3\dot{z} = u_{\text{ref}} - l_1\dot{\theta} - l_2K_2\theta - l_3\dot{z} = u_{\text{ref}} - Lx$$

with $L = (l_1, l_2K_2, l_3)$. The closed loop system becomes

$$\dot{x} = Ax + B(u_{\text{ref}} - Lx) = (A - BL)x + Bu_{\text{ref}}$$

The poles of the closed loop system are given by the eigenvalues of $A - BL$, i.e. the roots of the closed loop characteristic equation

$$\det(sI - (A - BL)) = \begin{vmatrix} s + K_1l_1 & K_1K_2l_2 & K_1l_3 \\ -1 & s & 0 \\ 0 & -K_2 & s \end{vmatrix}$$
$$= s^3 + K_1l_1s^2 + K_1K_2l_2s + K_1K_2l_3 = 0$$

The placement of all poles in −0.5 implies the following characteristic equation

$$(s+0.5)^3 = s^3 + 1.5s^2 + 0.75s + 0.125 = 0$$

One immediately obtains the solution

$$\begin{cases} l_1 = \dfrac{1.5}{K_1} \\ l_2 = \dfrac{0.75}{K_1K_2} \\ l_3 = \dfrac{0.125}{K_1K_2} \end{cases}$$

**9.5** The augmented system becomes

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

$$= \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{A_e} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x_e} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{B_e} u + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{B_r} r$$

We seek $L_e = \begin{pmatrix} l_1 & l_2 & l_3 \end{pmatrix}$ such that

$$\det(sI - (A_e - B_e L_e)) = (s+\alpha)(s^2 + 2\zeta\omega s + \omega^2)$$

Insertion of $A_e$, $B_e$ and $L_e$ into the above expression yields

$$s^3 + l_2 s^2 + l_1 s - l_3 \equiv s^3 + (\alpha + 2\zeta\omega)s^2 + (\omega^2 + 2\zeta\omega\alpha)s + \alpha\omega^2$$

Identifications of coefficients now yields

$$\begin{aligned} l_1 &= \omega^2 + 2\zeta\omega\alpha \\ l_2 &= \alpha + 2\zeta\omega \\ l_3 &= -\alpha\omega^2 \end{aligned}$$

**9.6** The estimation error $\tilde{x}$ fulfills $\dot{\tilde{x}} = (A - KC)\tilde{x}$. where $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}^T$. The characteristic equation of the estimation error becomes

$$\det(sI - (A - KC)) = s^2 + (4 + k_2)s + k_1 + 2k_2 + 3 = 0$$

The desired characteristic equation is

$$(s+4)^2 = s^2 + 8s + 16 = 0$$

Identification of coefficients yields $k_1 = 5$, $k_2 = 4$.

**9.7 a.** The characteristic equation of the closed loop system is given by

$$\det\big(sI - (A - BL)\big) = \begin{vmatrix} s + 4 + l_1 & 3 + l_2 \\ -1 & s \end{vmatrix} = s^2 + (4 + l_1)s + 3 + l_2 = 0$$

The desired characteristic equation is

$$(s+4)^2 = s^2 + 8s + 16 = 0$$

Which yields $l_1 = 4$ and $l_2 = 13$. The control law becomes

$$u = -l_1 x_1 - l_2 x_2 = -4x_1 - 13x_2$$

**b.** The states shall be estimated by means of a Kalman filter, i.e.

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

For $\tilde{x}$ we have

$$\frac{d\tilde{x}}{dt} = (A - KC)\tilde{x}$$

Determine $K$ such that all eigenvalues of the matrix $A - KC$ are placed in $\lambda = -6$.

$$\begin{aligned}\det(\lambda I - A + KC) &= \lambda^2 + (4 + k_1 + 3k_2)\lambda + 3 + 3k_1 + 9k_2 \\ &= (\lambda + 6)^2 = \lambda^2 + 12\lambda + 36\end{aligned}$$

Identify the coefficients and solve for $k_1$ and $k_2$:

$$\begin{cases} 4 + k_1 + 3k_2 = 12 \\ 3 + 3k_1 + 9k_2 = 36 \end{cases} \Rightarrow \begin{cases} k_1 + 3k_2 = 8 \\ k_1 + 3k_2 = 11 \end{cases}$$

The system of equations lacks solution, see the comment below.

**c.** The states are to be estimated by a Kalman filter, for which the eigenvalues of $A - KC$ shall be chosen such that

$$\lambda^2 + (4 + k_1 + 3k_2)\lambda + 3 + 3k_1 + 9k_2 = (\lambda + 3)^2 = \lambda^2 + 6\lambda + 9$$

Identifying coefficients and solving for $k_1$ and $k_2$ yields

$$\begin{cases} 4 + k_1 + 3k_2 = 6 \\ 3 + 3k_1 + 9k_2 = 9 \end{cases} \Rightarrow \begin{cases} k_1 + 3k_2 = 2 \\ k_1 + 3k_2 = 2 \end{cases}$$

This leaves only one equation, which implies that there exists infinitely many solutions, e.g. $k_1 = 2$, $k_2 = 0$ or $k_1 = 0$, $k_2 = \frac{2}{3}$ etc.

The drawback of the proposed observer pole placement is that it yields an estimation slower than the closed loop system. This does not affect the response to reference changes, which is governed by the poles of the closed loop system. However, the handling of process disturbances becomes slower.

*Comment.* An inspection of the system's observability shows that

$$\det W_o = \begin{vmatrix} C \\ CA \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & -3 \end{vmatrix} = 0$$

I.e. the system is not observable. The transfer function of the system is given by

$$G(s) = C(sI - A)^{-1}B = \frac{s+3}{s^2 + 4s + 3} = \frac{1}{s+1}$$

The eigenvalue $-3$ corresponds to a non-observable mode. The mode is, however, controllable, which follows from the canonical controllable form realization of the system. The characteristic equation of $A - KC$ can be written

$$\det(\lambda I - A + KC) = (\lambda + 3)(\lambda + k_1 + 3k_2 + 1)$$

This means that the Kalman filter has to estimate the non-observable mode with its own speed. I.e. (at least) one of the eigenvalues of $A - KC$ must be placed in $-3$. This explains the failure to compute a Kalman filter when the eigenvalues were to be placed in $-6$ and a success when they were to be placed in $-3$. Note that in cases such as this one, the result is either that there does not exist a solution $K$ to the Kalman filter problem, or that it exists infinitely many solutions. When the system is observable, there exists a unique solution $K$.

**9.8 a.**

$$\det(sI - (A - BL)) = \det \begin{pmatrix} s & -1 \\ l_1 & s + l_2 \end{pmatrix} = s^2 + l_2 s + l_1$$
$$\equiv s^2 + 2\zeta\omega s + \omega^2$$

I.e. $l_1 = \omega^2$, $l_2 = 2\zeta\omega$. The transfer function from $r$ to $y$ is given by

$$Y = \frac{1}{s^2 + l_2 s + l_1} l_r R$$

This transfer function has static gain 1 if

$$l_r = l_1 = \omega^2$$

**b.**

$$\det(sI - (A - KC)) = \det \begin{pmatrix} s + k_1 & -1 \\ k_2 & s \end{pmatrix} = s^2 + k_1 s + k_2$$
$$\equiv s^2 + 2as + a^2$$

which yields $k_1 = 2a$, $k_2 = a^2$.

**c.** We have

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) = (A - BL - KC)\hat{x} + Ky + Bl_r r \\ u &= -L\hat{x} + l_r r \end{aligned}$$

i.e. a system with inputs $r$ and $y$ and output $u$.

**d.**

$$\dot{x} = Ax + Bu = Ax - BL\hat{x} + Bl_r r$$
$$= Ax - BL(x - \tilde{x}) + Bl_r r = (A - BL)x + BL\tilde{x} + Bl_r r$$
$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu - K(y - C\hat{x}) = (A - KC)\tilde{x}$$

Thus

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - BL & BL \\ 0 & A - KC \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bl_r \\ 0 \end{pmatrix} r$$

The characteristic equation is given by

$$\det\left(sI - \begin{pmatrix} A - BL & BL \\ 0 & A - KC \end{pmatrix}\right)$$
$$= \det(sI - (A - BL))\det(sI - (A - KC))$$
$$= \left(s^2 + 2\zeta\omega s + \omega^2\right)\left(s^2 + 2as + a^2\right) = 0$$

**e.** The transfer function from $r$ to $y$ is given by

$$\begin{pmatrix} C & 0 \end{pmatrix}\left(sI - \begin{pmatrix} A - BL & BL \\ 0 & A - KC \end{pmatrix}\right)^{-1}\begin{pmatrix} Bl_r \\ 0 \end{pmatrix} =$$
$$= C(sI - (A - BL))^{-1}Bl_r$$

i.e. the same transfer functions when the states are measured directly rather tan estimated by means of a Kalman filter.

**9.9 a.** $L = \begin{pmatrix} 0 & 2\sqrt{2} \end{pmatrix}$, $K = \begin{pmatrix} 6 - 5/2 \end{pmatrix}^T$

**b.**

$$\dot{\hat{x}} = \begin{pmatrix} -6 & -2 \\ 9/2 & -2\sqrt{2} \end{pmatrix}\hat{x} + \begin{pmatrix} 6 \\ -5/2 \end{pmatrix} y$$
$$u = -\begin{pmatrix} 0 & 2\sqrt{2} \end{pmatrix}\hat{x}$$

**c.** According to the solution of sub-assignment 9.8c, the following equation is obtained

$$(s^2 + 2\sqrt{2} + 4)(s^2 + 6s + 9) = 0$$

**d.**

$$G_{KF} = \sqrt{2}\frac{5s - 24}{s^2 + (6 + 2\sqrt{2})s + 12\sqrt{2} + 9}$$

**9.10**

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the observability matrix is non-singular (has full rank), i.e. the system is observable.

$$\det(sI - (A - KC)) = s^3 + k_1 s^2 + k_2 s + k_3 = (s + \alpha)\left(s^2 + 2\zeta\omega s + \omega^2\right)$$

Identification of coefficients yields $k_1 = \alpha + 2\zeta\omega$, $k_1 = \omega^2 + 2\alpha\zeta\omega$ and $k_3 = \alpha\omega^2$. The equations of the Kalman filter (the observer) become

$$\dot{\hat{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} (y - \hat{x}_1) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u$$

Note that $\hat{x}_3$, our estimation of the zero error, is a pure integration of $y - \hat{x}_1$.

# Solutions to Chapter 10. Controller Structures

**10.1** The disturbance $d$ does obviously lack influence if

$$G_1(s)H(s) + 1 = 0 \quad \Leftrightarrow \quad H(s) = -\frac{1}{G_1(s)}$$

To be a practically useful control law it is required that the disturbance can be measured, that *the model* $G_1(s)$ of the heating system is a "good" description of reality and that the inverse transfer function $1/G_1(s)$ is practically realizable. This means that $H(s)$ must not contain derivatives of the signal $d$. The realization of $H(s)$ can also be problematic if $G_1(s)$ lacks a stable inverse (i.e. if $G_1(s)$ has right half plane zeros, which is equivalent to being a non-minimum phase system). Further, we cannot invert processes with low pass characteristics more than at low frequencies and delays can obviously not be inverted.

**10.2** A block diagram for the system is shown in figure 10.1. Mass balance

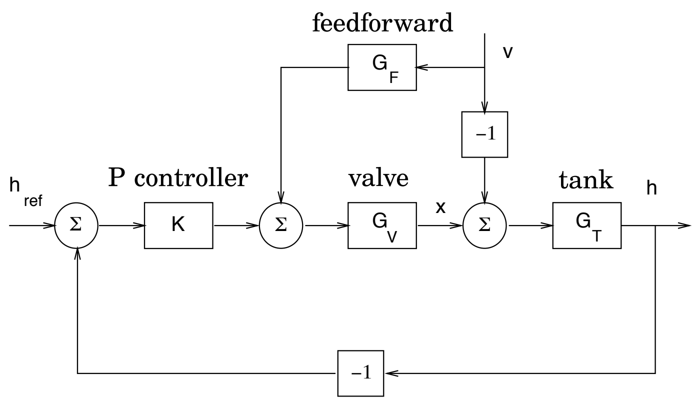



**Figure 10.1** Block diagram of the level controlling system in assignment 10.2.

for the tank yields

$$A\frac{dh}{dt} = x(t) - v(t)$$

Laplace transformation gives ($A = 1\ m^2$)

$$H(s) = \frac{1}{s}(X(s) - V(s))$$

The transfer function of the tank is thus

$$G_T(s) = \frac{1}{s}$$

**a.** The closed loop transfer function becomes

$$G(s) = \frac{G_T G_V K}{1 + G_T G_V K} = \frac{K}{0.5s^2 + s + K}$$

The characteristic polynomial is hence

$$s^2 + 2s + 2K$$

The desired characteristic polynomial is

$$(s + \omega)^2 = s^2 + 2\omega s + \omega^2$$

Identification of coefficients yields

$$\begin{cases} \omega = 1 \\ K = \dfrac{1}{2} \end{cases}$$

The transfer function form $v(t)$ to $h(t)$ is given by

$$H(s) = -\frac{G_T}{1 + G_T G_V K} V(s) = -\frac{1 + 0.5s}{s(1 + 0.5s) + K} V(s)$$

If $v(t)$ is a step of amplitude 0.1 we obtain $V(s) = 0.1/s$. The final value theorem gives

$$h(\infty) = \lim_{s \to 0} sH(s) = -\frac{0.1}{K}$$

given that the limit exists and that the final value theorem is applicable.

**b.** A PI controller has the transfer function

$$G_R(s) = K(1 + \frac{1}{sT_i})$$

The closed loop transfer function becomes

$$G(s) = \frac{G_T G_V G_R}{1 + G_T G_V G_R} = \frac{K(1 + sT_i)}{s(1 + 0.5s)sT_i + K(1 + sT_i)}$$

The characteristic polynomial becomes

$$s^3 + 2s^2 + 2Ks + \frac{2K}{T_i}$$

The desired characteristic polynomial is

$$(s + \omega)^3 = s^3 + 3\omega s^2 + 3\omega^2 s + \omega^3$$

Identification of coefficients yields

$$\begin{cases} \omega = \dfrac{2}{3} \\ K = \dfrac{2}{3} \\ T_i = \dfrac{9}{2} \end{cases}$$

**c.** The relation between the flow disturbance $v$ and the level $h$ is given by

$$H(s) = \frac{G_T(G_V G_F - 1)}{1 + G_T G_V G_R} V(s)$$

To eliminate the influence of $v$, we choose

$$G_F(s) = \frac{1}{G_V} = 1 + 0.5s$$

**10.3** The closed loop system has the transfer function

$$\frac{(G_R + K_f)G_P}{1 + G_P G_R} = \frac{(K + K_f)s + K/T_i}{s^2 + (3 + K)s + K/T_i}$$

**a.** The characteristic equation of the closed loop system is

$$s^2 + (3 + K)s + K/T_i = 0$$

The desired characteristic equation is

$$(s + 2 - 2i)(s + 2 + 2i) = s^2 + 4s + 8 = 0$$

Identification of coefficients yields $K = 1$ and $T_i = 1/8$.

**b.** The feedforward $K_f$ affects the zeros of the closed loop system, but leaves the poles unaffected. The poles can be placed by means of the controller $H$ in order to obtain adequate supression of disturbances, cf. sub-assignment a above.

One can subsequently translate the zeros by means of $K_f$ in order to e.g. reach a desired overshoot in the reference step responses.

The zero of the closed loop system is eliminated by choosing $K_f = -K$. With the pole placement in sub-assignment a, which corresponds to a relative damping $\zeta = 1/\sqrt{2} \approx 0.7$, the overshoot of the closed loop system becomes approximately 5%.

**10.4** The block diagram in assignment 10.3 can be re-drawn according to figure 10.2. By comparing to the block diagram in assignment 10.3 we see that $H_{ff} = H + K_f$ and $H_{fb} = H$. Observe that manipulation of $K_f$ offers the possibility to neutralize the derivation in $H$, i.e. achieve a controller which derivates the output, but not the reference value.

**10.5** The system has three inputs: the reference $y_r$ and the two disturbances $v_1$ and $v_2$. The transfer functions between these three signals and the output $y$ are given by

$$Y = \frac{G_1 G_2 G_{R1} G_{R2}}{1 + G_1 G_{R1} + G_1 G_2 G_{R1} G_{R2}} Y_r + \frac{G_1 G_2}{1 + G_1 G_{R1} + G_1 G_2 G_{R1} G_{R2}} V_1$$
$$+ \frac{(1 + G_1 G_{R1}) G_2}{1 + G_1 G_{R1} + G_1 G_2 G_{R1} G_{R2}} V_2$$

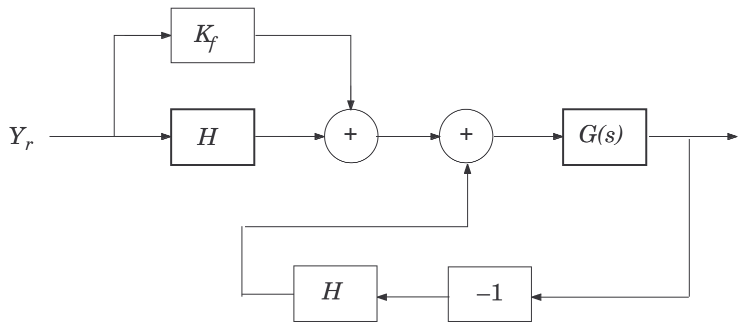



**Figure 10.2** Modified block diagram in assignment 10.3.

Let us call the three transfer functions $G_{yr}$, $G_{v1}$ and $G_{v2}$, respectively. Ideally we would have $G_{yr} = 1$ and $G_{v1} = G_{v2} = 0$ for all frequencies. This is, however, not achievable. Nonetheless, we can assure that it holds in stationarity, i.e. for $s = 0$. For a P controller we have $G_R(0) = K$, where $K$ is the gain of the controller. For a PI controller it holds that $G_R(0) = \infty$.

The transfer function $G_{yr}$ becomes unity if $G_{R2}$ is a PI controller. The transfer function $G_{v1}$ becomes 0 if any of the controllers are PI. The transfer function $G_{v2}$, however, is only zero if $G_{R2}$ is a PI controller.

Consequently $G_{R2}$ must contain an integral part in order to guarantee 0 stationary control error. The controller $G_{R1}$ can then be chosen to be a P controller. (If we furthermore want the internal signal $y_1$ to coincide with its reference, also this controller would need an integral part.)

**10.6a.** The closed loop transfer function is given by

$$G_{\text{inner}}(s) = \frac{K_1 G_1(s)}{1 + K_1 G_1(s)} = \frac{2K_1}{s + 2 + 2K_1}$$

In order to make the system 5 times as fast, the pole of the closed loop system must be placed in $s = -10$, calling for $K_1 = 4$.

**b.** The approximation $G_{\text{inner}}(s) \approx G_{\text{inner}}(0) = 0.8$ yields

$$G_{\text{outer}}(s) = \frac{G_{R2}(s) G_2(s) G_{\text{inner}}(0)}{1 + G_{R2}(s) G_2(s) G_{\text{inner}}(0)} = \frac{(K_2 s + \frac{K_2}{T_i})0.8}{s^2 + 0.8K_2 s + 0.8\frac{K_2}{T_i}}$$

The specification of a system 10 times slower than the inner loop calls for a pole in $s = -1$. Since we deal with a second order system, we choose to locate both poles in $s = -1$ (somewhat slower than the single pole case). This yields $K_2 = 2.5$ and $T_i = 2$.

*Comment.* A general rule when cascading controllers is to make the inner loop 5–10 times faster than the outer loop in order to enable separation of the controller calculations for the two loops. The actual closed loop system (without approximations) becomes

$$G_{\text{outer}}(s) = \frac{10(2s+1)}{s^3 + 10s^2 + 20s + 10}$$

and has poles in approximately −7.516, −1.702 and −0.7815 where the slower pole ($s = -0.7815$) will be the one essentially determining the speed of the system. This corresponds fairly well to the specified speed.

**10.7a.** Since the steam flow is assumed to be constant, we can let $F = 0$, which yields the following description of the dome

$$Y(s) = \frac{10^{-3}}{s} M(s)$$

Since the controller is of P type we have $M(s) = K(Y_r - Y)$, where $Y_r$ denotes the reference dome level. This yields

$$Y(s) = \frac{K}{K + 10^3 s} Y_r(s)$$

Since the system is linear and subject to negative feedback, a step disturbance in the level gives rise to the same transient behavior as a step disturbance in the reference. Hence let $Y_r(s) = \frac{1}{s}$. Inverse transformation of $Y(s)$ yields

$$y(t) = 1 - e^{-K10^{-3}t}$$

The specification on the settling time of the system now yields

$$y(10) = 1 - e^{-K10^{-2}} = 0.9 \quad \Rightarrow \quad K = 230$$

**b.** The dome and P controller are described by

$$Y(s) = \frac{K}{K + 1000s} Y_r(s) + \frac{s - 0.01}{(s+0.1)(1000s + K)} F(s)$$

Let $Y_r(s) = 0$. A step disturbance in the steam flow $F(s) = \frac{1}{s}$ thus gives

$$Y(s) = \frac{s - 0.01}{(s+0.1)(1000s+K)} \frac{1}{s}$$

The final value theorem yields

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{s-0.01}{(s+0.1)(1000s+K)}\frac{1}{s} = \frac{-0.1}{K}$$

and stationary error becomes

$$e = y_r - y = -y = \frac{0.1}{K}$$

**c.** Determine a feedforward link $H(s)$ from steam flow $F(s)$ to feed water flow $M(s)$ for the initial system, such that the level $Y(s)$ becomes independent of changes in the steam flow.

The system with feedforward $H(s)F(s)$ is described by

$$\begin{aligned} Y(s) &= \frac{10^{-3}}{s}(M(s) + H(s)F(s)) + \frac{s-0.01}{s(s+0.1)}10^{-3}F(s) \\ &= \frac{10^{-3}}{s}M(s) + \frac{10^{-3}}{s}\left(\frac{s-0.01}{s+0.01} + H(s)\right)F(s) \end{aligned}$$

We want the influence from $F(s)$ to be zero. Therefore choose $H(s)$ so that the expression in front of $F(s)$ becomes zero. This criterion is fulfilled when

$$H(s) = -\frac{s-0.01}{s+0.1}$$

which gives the desired feedforward.

**10.8** The transfer function $d \to y$ is

$$(1 + H(s)G_1(s))\, G_2(s) = \left(1 + H(s)\frac{1}{s+1}\right)\frac{1}{s}$$

It is apparently $\equiv 0$ if we choose $H(s) = H_1(s) = -1/G_1(s) = -(s+1)$. Unfortunately, $H_1(s)$ is not realizable since it contains derivations.

Try instead

$$H(s) = H_2(s) = -\frac{(s+1)}{(sT+1)}$$

where $T$ is "small". Now $H_2(s)$ approximates $H_1(s)$ 'well' for 'low' frequencies. From the problem formulation it is evident that 'well' shall be interpreted as 'with an error of at most 10%' and 'low frequencies' means $|\omega| \le 5$. With $H(s) = H_2(s)$, the transfer function from $d$ to $y$ becomes

$$G(s) = -\frac{T}{1+sT}$$

Now choose $T$ such that $|G(i\omega)| \le 0.1$ for $|\omega| \le 5$.

Since $|G(i\omega)|$ is declining for $\omega > 0$ it is sufficient that $|G(0)| = T \le 0.1$, i.e. that $T \le 0.1$.

**10.9** The delay margin is given by

$$L_m = \frac{\varphi_m}{\omega_c}$$

First we compute the cross-over frequency $\omega_c$ as

$$|G_0(i\omega_c)| = |G_P(i\omega_c)G_R(i\omega_c)| = \left|\frac{2}{i\omega_c(i\omega_c+1)}\right| = \frac{2}{\omega_c\sqrt{\omega_c^2+1}} \equiv 1$$

$$\Leftrightarrow \quad \omega_c^4 + \omega_c^2 - 4 = 0 \quad \Leftrightarrow \quad \omega_c = \sqrt{\frac{-1+\sqrt{17}}{2}} = 1.25$$

Then we calculate the the phase margin $\varphi_m$

$$\varphi_m = \pi + \arg G_0(i\omega_c) = \pi - \frac{\pi}{2} - \arctan \omega_c = 0.675$$

We thus obtain $L_m = \varphi_m/\omega_c = 0.54$.

**10.10a.** The one second delay $e^{-s}$ is considered part of the process.

| | |
|---|---|
| *Controller* | $G_R(s) = K$ |
| *Process* | $G_P(s) = \dfrac{1}{s(s+1)}e^{-s}$ |
| *Model* | $\hat{G}_P(s) = G_P(s) = \dfrac{1}{s(s+1)}e^{-s}$ |
| *Model without delay* | $\hat{G}_P^0(s) = \dfrac{1}{s(s+1)}$ |

**b.** According to the block diagram the control signal is given by

$$U(s) = G_R(s)\Big(E(s) + \hat{G}_P(s)U(s) - \hat{G}_P^0(s)U(s)\Big)$$

The transfer function of the controller becomes

$$\begin{aligned} U(s) &= \frac{G_R(s)}{1 - G_R(s)\hat{G}_P(s) + G_R(s)\hat{G}_P^0(s)}E(s) \\ &= \frac{2}{1 - \dfrac{2}{s(s+1)}e^{-s} + \dfrac{2}{s(s+1)}}E(s) = \frac{2s(s+1)}{s(s+1)+2-2e^{-s}}E(s) \end{aligned}$$

The Bode plot of the controller is shown in figure 10.3. One notes that the Smith predictor gives a large phase lead at the cross-over frequency of the initial system.

**c.**

$$\begin{aligned} U(s) &= \frac{2s(s+1)}{s(s+1)+2-2e^{-s}}E(s) \approx \frac{2s(s+1)}{s(s+1)+2-2(1-s)}E(s) \\ &= \frac{2(s+1)}{s+3}E(s) \end{aligned}$$

This is a phase lead link with $N = 3$.

**10.11** The gain curve of the system is given by

$$|G(i\omega)| = \frac{k}{\omega}$$

It is sufficient to read the value of the gain curve at a single frequency in order to determine $k$. The gain is e.g. 1 at approximately $\omega = 4.5$. This yields

$$1 = \frac{k}{4.5} \quad \Leftrightarrow \quad k = 4.5$$

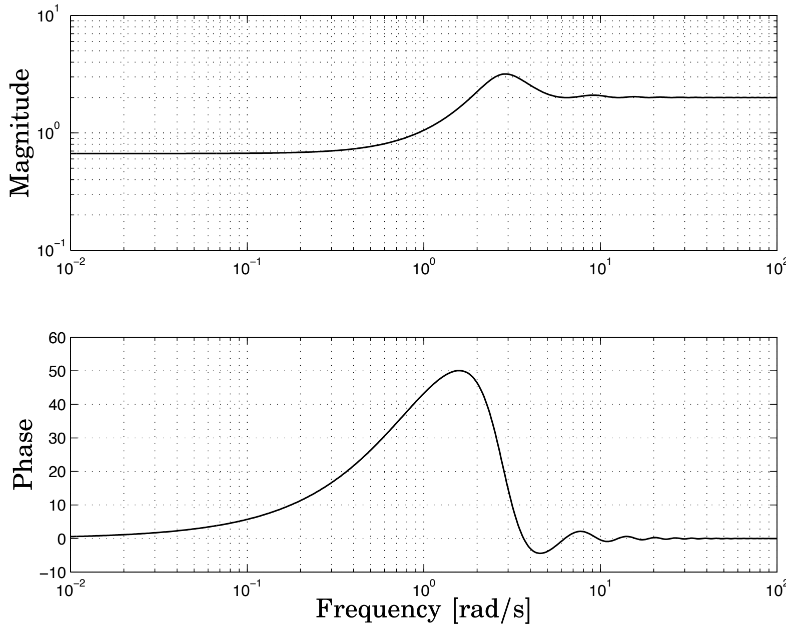



**Figure 10.3** Bode plot of the Smith predictor.

The phase curve of the system is given by

$$\arg G(i\omega) = -\pi/2 - \omega L$$

Now, it is sufficient to read the value of the phase curve at a single frequency in order to determine $L$. The phase is e.g. $-\pi$ at approximately $\omega = 120$. This yields

$$-\pi = -\pi/2 - 120L \quad \Leftrightarrow \quad L = 0.013$$

# Solutions to Chapter 11. Design Examples

**11.1a.** The phase curve for $v = 3$ knots cuts $-180°$ at $\omega_o \approx 0.03$ rad/s. At this frequency we have $|G(i0.03)| \approx 2$. The gain $K$ must hence be smaller than 0.5 in order to yield a stable closed loop system.

**b.** In order to acquire the cross-over frequency $\omega_c$ and phase margin $\varphi_m$ it is required that

$$\begin{cases} |G_r(i\omega_c)G(i\omega_c)| = 1 \\ \arg G_r(i\omega_c)G(i\omega_c) = \varphi_m - 180° \end{cases}$$

where $G_r(s) = K(1 + T_D s)$. This leads to the equations

$$\begin{cases} K|G(i\omega_c)|\sqrt{1 + T_D^2\omega_c^2} = 1 \\ \arg G(i\omega_c) + \arctan T_D\omega_c = \varphi_m - 180° \end{cases}$$

With $\omega_c = 0.03$ rad/s, $\varphi_m = 60°$, $|G(i\omega_c)| \approx 2$ and $\arg G(i\omega_c) \approx -180°$ we obtain

$$\begin{cases} T_d = \dfrac{\tan 60°}{0.03} = \dfrac{\sqrt{3}}{0.03} \approx 57.7 \\ K = \dfrac{1}{|G(i\omega_c)|\sqrt{1 + T_d^2\omega_c^2}} \approx \dfrac{1}{2 \cdot 2} = 0.25 \end{cases}$$

**c.** If the speed suddenly increases from 3 to 7 knots, we have to turn to the dotted Bode-curves in figure **??**. The most drastic change is that the gain curve has been raised by a factor 20. Additionally, the phase curve has decreased for frequencies above 0.03 rad/s. This results in heavily reduced phase- and gain margins. A more thorough examination shows that this in fact leads to instability of the closed loop system. This can be seen in the Bode plot in figure 11.1, which shows both the nominal case $v = 3$ knots and the case $v = 7$ knots. One way to avoid this problem is to instead choose $v = 7$ knots

as the nominal case for the calculation of the PD controller. This, however, means that one has to accept a slower settling time for the slowest speed $v = 3$ knots. A better way is to let $K$ and $T_d$ depend on the speed $v$. This method is known as *gain scheduling*.

**d.** The transfer function from $\beta$ to $h$ can be approximated by

$$G_{h\beta}(s) = \frac{k_v v}{s^3}$$

From the Bode plot one sees that $|G_{h\beta}(i \cdot 0.1)| \approx 0.04$ for $v = 3$ knots $= 3 \cdot 1.852/3.6 \approx 0.5144 \cdot 3$ m/s, which yields

$$k_v \approx \frac{0.1^3 \cdot 0.04}{3 \cdot 0.5144} \approx 2.6 \cdot 10^{-5}$$

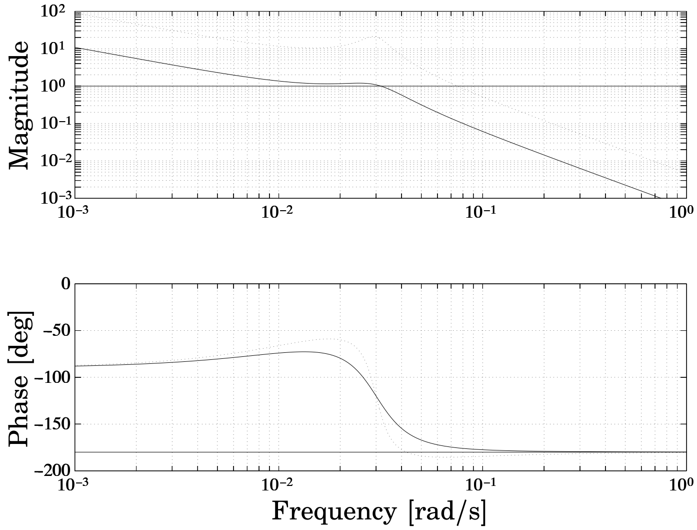



**Figure 11.1** Bode plot of the PD compensated open loop system in assignment 11.1. The solid curves show the case $v = 7$ knots. Note that the latter case yields an unstable closed loop system.

**e.** The characteristic equation of the closed-loop is given by

$$s^3 + Kk_v v = 0$$

Since not all coefficients are positive, the closed-loop system is not asymptotically stable for any value of $K$. In sub-assignment a it was concluded through the measured frequency response that the closed loop system was stable for $K < 0.5$. The explanation to this apparent contradiction is found in the Bode plot which was used in sub-assignment a: The approximation only holds for high frequencies ($\omega > 0.05$). For low gains, such as $K < 0.5$, the cross-over frequency $\omega_c < 0.03$ lies outside the valid range of the model.

For e.g. $\omega < 0.03$ the Bode plot shows a phase above $-180°$ while the simplified model features the phase $-270°$ for *all* frequencies.

**f.** If $x = (\dot{\alpha}, \alpha, h)^T$ and $u = \beta$ we obtain the state space equations

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & v & 0 \end{pmatrix} x + \begin{pmatrix} k_v \\ 0 \\ 0 \end{pmatrix} u$$

With the state feedback $u = -Lx + u_r$ the characteristic polynomial of the closed loop system becomes

$$p(s) = \det(sI - A + BL) = s^3 + k_v l_1 s^2 + k_v l_2 s + k_v v l_3$$

the desired characteristic polynomial is

$$p(s) = (s+\gamma\omega_0)(s^2+2\zeta\omega_0 s+\omega_0^2) = s^3+(\gamma+2\zeta)\omega_0 s^2+(2\gamma\zeta+1)\omega_0^2 s+\gamma\omega_0^3$$

Direct comparison gives

$$\begin{cases} l_1 = \dfrac{(\gamma + 2\zeta)\omega_0}{k_v} \\ l_2 = \dfrac{(2\gamma\zeta + 1)\omega_0^2}{k_v} \\ l_3 = \dfrac{\gamma\omega_0^3}{k_v v} \end{cases}$$

**g.** Here stationarity means constant height, $h = h_{\text{ref}}$. This in turns mean that all derivatives of $h$ must be zero, i.e. $\alpha = 0$ and $\dot{\alpha} = 0$. When the height has reached its correct value the control signal $u = \beta$ must also be zero since the submarine would otherwise continue to rise. Thus $L_r$ is obtained from the equation

$$0 = L_r h_{\text{ref}} - l_1 \cdot 0 - l_2 \cdot 0 - l_3 h_{\text{ref}}$$

For $v = 3$ knots, we end up with the following result

$$L_r = l_3 = \frac{\gamma\omega_0^3}{k_v v} \approx \frac{\gamma\omega_0^3 \cdot}{2.6 \cdot 10^{-5} \cdot 3 \cdot 0.5144} \approx \frac{\gamma\omega_0^3 \cdot}{4.0 \cdot 10^{-5}}$$

**h.** At a momentary disturbance $\Delta h = 0.1$ m the rudder angle becomes

$$\Delta\beta = l_3 \cdot \Delta h = \frac{\gamma\omega_0^3}{v k_v} \cdot 0.1 \approx \frac{0.2\omega_0^3}{3 \cdot 0.5144 \cdot 2.6 \cdot 10^{-5}}$$

Since $\Delta\beta \leq 5°$, we must have

$$\omega_0 \leq \left( \frac{5 \cdot 3 \cdot 0.5144 \cdot 2.6 \cdot 10^{-5}}{0.2} \right)^{\frac{1}{3}} \approx 0.1$$

**11.2a.** The oscillation frequency $\omega_o \approx 27$ rad/s and critical gain $K_c \approx 3.6$ can be read from the Bode plot. The oscillation period is hence $T_o = 2\pi/\omega_o \approx 0.23$. This yields the PID parameters $K = 0.6K_c \approx 2.2$, $T_i = T_o/2 \approx 0.12$ and $T_d = T_o/8 \approx 0.03$. The step response of the closed loop system is shown in figure 11.2. The specifications are

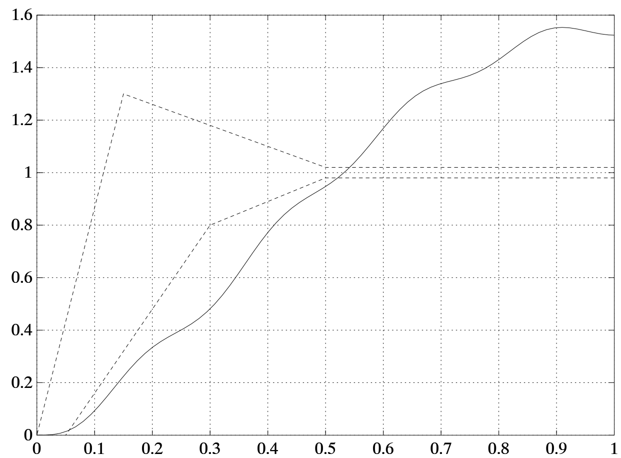



**Figure 11.2** The step response with PID control according to Ziegler-Nichols.

apparently not fulfilled. A PID controller (with filter factor) can be considered a second order controller with integral action. As a matter of fact, the specifications can be met, using a more general second order controller with integral action (see figure 11.3). If one tries to interpret it as a PID controller, one would end up with a negative derivative time $T_d$.

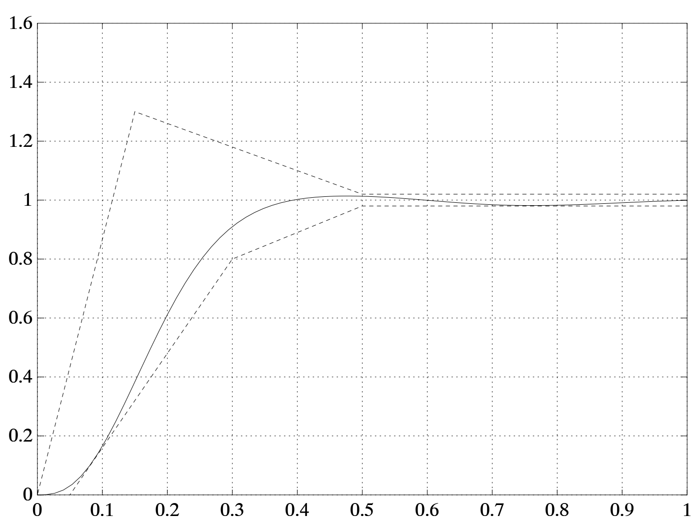



**Figure 11.3** The step response of the closed loop system with a second order integrating controller.

**b.** In stationarity all derivatives of the states must be zero $\dot{x} = 0$. It hence holds that

$$\begin{cases} 0 = Ax^o + Bu^o = (A - BL)x^o + BL_r y_r \\ y_r = y^o = Cx^o \end{cases}$$

This yields

$$L_r = -\frac{1}{C(A - BL)^{-1}B}$$

**c.** With $x$ augmented to $x_e = (x_1, x_2, x_3, x_i)^T$ we obtain

$$\dot{x}_e = \begin{pmatrix} -\frac{d_1+d_f}{J_1} & \frac{d_f}{J_1} & -\frac{k_f}{J_1} & 0 \\ \frac{d_f}{J_2} & -\frac{d_f+d_2}{J_2} & \frac{k_f}{J_2} & 0 \\ 1 & -1 & 0 & 0 \\ 0 & k_{\omega_2} & 0 & 0 \end{pmatrix} x_e + \begin{pmatrix} \frac{k_m k_i}{J_1} \\ 0 \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} y_r$$

$$y = \begin{pmatrix} 0 & k_{\omega_2} & 0 & 0 \end{pmatrix} x_e$$

where the reference $y_r$ has been introduces as an extra input.

**d.** The approximate value of $\omega_m$ becomes

$$\omega_m \approx -\frac{\ln 0.02}{0.5 \cdot 0.38} \approx 20$$

**e.** When it comes to load disturbances, the fast Kalman filter ($\omega_o = 40$) has the best performance. It is also best when it comes to suppressing the influence of of measurement noise. However, it is the worst choice when it comes to suppressing the influence of noise in the control signal. The two cases $\omega_o = 10$ and $\omega_o = 20$ feature approximately the same noise sensitivity, while $\omega_o = 10$ is slower when it comes to eliminating load disturbances. A satiable choice is thus $\omega_o = 20$.