# Lecture 1

- Course contents
- Practical stuff - book - today pp. 71-101
- Math background
- Laplace transform AK 17
- Transient and initial states AK 18
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# Course content

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|---|---|
| Lec1 | Basic system theory |
| Lec2 | Argument variation principle, Nyquist theorem, Bode's relations |
| Lec3 | Stability, Robustness, Sensitivity Function |
| | w7 Handin 1: Laplace transform and Frequency plots. |
| Lec 4 | State coordinate change, zeros, state feedback, observers |
| Lec5 | Controllability and Observability, Kalman's decomposition theorem |
| Lec6 | Linear mappings and least squares problems |
| | w10: HANDIN 2: State representations |

Presentations HANDIN 1: TBD; no presentations HANDIN2

# Math Background - from Spanne's 'blixtkurs'

- $\int_C f(z)dz$, $C : \{z(t), t \in [a, b]\}$, $\int_a^b f(z)\frac{dz}{dt}dt$,
- important example:$f(z) = \frac{1}{z-p}$, with $C : \{z(t) = p + re^{it}$, $t \in [0, 2\pi]\}$
- $f(z)$ analytic, closed curve, Cauchy's integral theorem: different paths same integral, deformation of integration path
- $\int_C \frac{f(z)}{z-p}dz = f(p)2\pi i$, Cauchy's integral formula
- $\{p_k\}_1^n$ poles to $f(z)$ inside C, then $\int_C = \int_{C_1} + \cdots + \int_{C_n}$, $\text{Res}_{z=p_k} f(z) = \frac{1}{2\pi i}\int_{C_k} f(z)dz$, residue calculus

# Laplace transform

- Double vs single sided Laplace
- Strip of definition. Different for different signals
- Transfer functions. How do we handle different strips of definition?
- Use one sided transforms + analytic continuation
- Makes it possible to also analyse unstable causal systems

# Laplace transform - definition - convergence

*Double-sided:* Consider time functions $f(t)$, $-\infty < t < \infty$

$$F(s) = (\mathcal{L}_{II} f)(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

Convergens in strip $\Omega:\ \alpha < Re\ s < \beta$, $\quad F(s)$ analytic in $\Omega$.

$e^{-\alpha t} f(t) \to 0, \quad t \to \infty,$ och $\quad e^{-\beta t} f(t) \to 0, \quad t \to -\infty.$

Ex $\alpha < 0$ and $\beta > 0$ requires exponential convergence for both $t \to \infty$ and $t \to -\infty$.

*Single-sided:* Consider $f(t)$, $0 \leq t < \infty$

$$F(s) = (\mathcal{L}_{I} f)(s) = \int_{0}^{\infty} e^{-st} f(t) dt$$

Converges in half plane $\Omega:\ \alpha < Re\ s$, $\quad F(s)$ analytic in $\Omega$.

$e^{-\alpha t} f(t) \to 0, \quad t \to \infty$, note $\alpha > 0$ allows $f(t) \to \infty, \quad t \to \infty$.

# Laplace transform - example

$$f(t) = e^{2t},\ t \geq 0, \quad F = \mathcal{L}_I\{f\}, \quad F(s) = \lim_{T\to\infty} \int_0^T e^{2t} e^{-st} dt$$

$$F(s) = \lim_{T\to\infty} \left[ \frac{1}{2-s} e^{(2-s)t} \right]_0^T = \frac{1}{2-s} \lim_{T\to\infty} \left\{ e^{(2-s)T} - 1 \right\}$$

$$\lim_{T\to\infty} e^{(2-s)T} = 0, \quad Re\, s > 2$$

So

$$F(s) = \frac{1}{s-2}, \quad Re\, s > 2$$

Extend domain of definition with analytic continuation to $\mathbf{C} - \{s = 2\}$, only possible such function is $F(s) = \frac{1}{s-2}$

Nice video about analytic continuation:
www.youtube.com/watch?v=sD0NjbwqlYw&t=3s

# Transfer functions for causal systems

Weight function

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau$$

$$h(\tau), \quad 0 \le \tau < \infty$$

$$G(s) = (\mathcal{L}_I h)(s)$$

$$Y(s) = G(s)U(s)$$

# Laplace transform relations

$$\mathcal{L}_I\left(f'\right) = sF(s) - f(0)$$

Proof:

$$\begin{aligned}\mathcal{L}_I\left(\frac{df}{dt}\right) &= \int_0^\infty e^{-st}\frac{df}{dt}dt \qquad (*) \\ &= s\int_0^\infty e^{-st}f(t)dt + \left[e^{-st}f(t)\right]_{t=0}^{\infty} = \\ &= sF(s) - f(0)\end{aligned}$$

(If both integrals converge and if $e^{-st}f(t) \to 0$ as $t \to \infty$).

# Quiz

What is $\mathcal{L}_I\left(f''\right)$?

a $s^2F(s) - f(0)$
b $s^2F(s) - f'(0)$
c $s^2F(s) - sf(0) - f'(0)$
d $s^2F(s) - sf'(0) - f(0)$

# Final Value Theorem - sketch

When $s \to 0$ in $(*)$ we get

$$\int_0^\infty \frac{df}{dt} dt = \lim_{s \to 0} sF(s) - f(0)$$

If the limit value $\lim_{t \to \infty} f(t)$ exists, then this can be written

$$\lim_{t \to \infty} f(t) - f(0) = \lim_{s \to 0} sF(s) - f(0)$$

which is the final value theorem

# Initial Value Theorem - sketch

If we instead let $s \to \infty$ we have

$$\lim_{s\to\infty} \int_0^\infty e^{-st} \frac{df}{dt} dt = \lim_{s\to\infty} sF(s) - f(0)$$

This motivates that we should have

$$0 = \lim_{s\to 0+} sF(s) - f(0)$$

which is the initial value theorem

Both the final and initial value theorems need conditions to guarantee that the calculations we just did are correct.

# Initial Value Theorem - sketch

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Both the final and initial value theorems need conditions to guarantee that the calculations we just did are correct.

# Initial and Final-value theorems - rational $F$

**Initial Value Theorem** Assume the Laplace transform $F(s)$ is rational and strictly proper. Then

$$\lim_{t \to +0} f(t) = \lim_{s \to +\infty} sF(s)$$

**Final Value Theorem.** Assume that $F(s)$ is rational and all poles to $sF(s)$ have negativ real part, then

$$\lim_{t \to +\infty} f(t) = \lim_{s \to +0} sF(s)$$

Sketch for rational $F(s)$: The theorem is true if $F(s) = (s-p)^k$ (check). Write $F$ as a sum of such terms.

# Proof slightly more general final value theorem

Om $\dfrac{f(t)}{e^{at}t^k} \to C \in \mathbb{R}$ då $t \to \infty$

så är $g(t) := \dfrac{f(t)}{e^{at}t^k}$ begränsad för stora $t$

$$F(s) = \int_0^\infty e^{-st}\, e^{at}\, t^k\, g(t)\, dt = \begin{bmatrix} x = (s-a)t \\ s > a \\ s \text{ reell} \end{bmatrix}$$

$$= \int_0^\infty e^{-x} \frac{x^k}{(s-a)^k}\, g\!\left(\frac{x}{s-a}\right) \frac{dx}{s-a}$$

$$\lim_{s \to a^+} (s-a)^{k+1} F(s) = \lim_{s \to a^+} \int_0^\infty e^{-x} x^k\, g\!\left(\frac{x}{s-a}\right) dx$$

$$= \lim_{t \to \infty} g(t) \int_0^\infty e^{-x} x^k\, dx$$

$$= \lim_{t \to \infty} \frac{f(t)}{e^{at}t^k}\, \Gamma(k+1)$$

Krav: $k+1 > 0$

# Transients and initial conditions

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0$$
$$y = Cx + Du$$

Laplace transform gives

$$sX(s) - x_0 = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1}(BU(s) + x_0)$$
$$Y = \underbrace{[C(sI - A)^{-1}B + D]}_{G(s)} U(s) + C(sI - A)^{-1}x_0$$

# Example: Sinusoidal input signal

$$\dot{x} = -x + u \qquad x(0) = x_0 \qquad u(t) = \sin t$$

gives after Laplace transform

$$sX(s) - x(0) = -X(s) + U(s), \qquad U(s) = \frac{1}{s^2+1}$$

Solving for $X$ gives

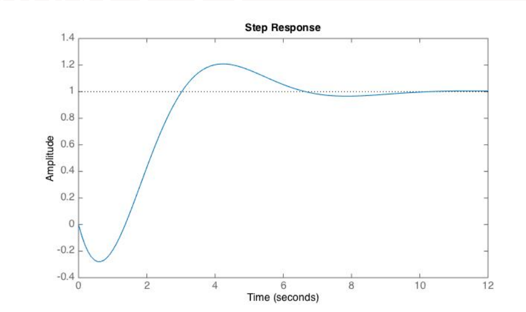
$$\begin{aligned} X(s) &= \frac{1}{s+1}(U(s) + x_0) = \frac{1}{s+1}\left(\frac{1}{s^2+1} + x_0\right) \\ &= \frac{0.5 - 0.5s}{s^2+1} + \frac{0.5 + x_0}{s+1} \end{aligned}$$

Invers transformation (table) gives

$$x(t) = \frac{1}{2}\sin t - \frac{1}{2}\cos t + (x_0 + \frac{1}{2})e^{-t}$$

# Laplace transform in Matlab (or Maple)

```
>> s=tf('s')
>> G = (1-s)/(s^2+s+1)
G =
     -s + 1
  -----------
  s^2 + s + 1
>> step(G)
```



# Laplace transform in Matlab (or Maple)

```
>> clear s
>> syms s t x0

>> ilaplace((1-s)/(s^2+s+1))
ans =
-exp(-t/2)*(cos((3^(1/2)*t)/2) - 3^(1/2)*sin((3^(1/2)*t)/2

>> ilaplace((0.5-0.5*s)/(s^2+1) + (0.5+x0)/(s+1))
ans =
sin(t)/2 - cos(t)/2 + exp(-t)*(x0 + 1/2)

>> latex(ans)
```

$$\frac{\sin(t)}{2} - \frac{\cos(t)}{2} + \mathrm{e}^{-t}\,\left(\mathrm{x0} + \frac{1}{2}\right)$$

# A sliding block - where will it stop?

A block is sliding according to

$$\ddot{y}(t) + c\dot{y}(t) = 0 \tag{1}$$

with start in position $y(0) = a$ and speed $\dot{y}(0) = b$. Determine $\lim_{t\to\infty} y(t)$.

Laplace transform of (1) gives

$$s^2 Y(s) - sy(0) - \dot{y}(0) + c[sY(s) - y(0)] = 0$$

$$Y(s) = \frac{sy(0) + \dot{y}(0) + cy(0)}{s^2 + cs}$$

Final value theorem gives

$$\lim_{t\to\infty} y(t) = \lim_{s\to+0} sY(s) = \lim_{s\to+0} \frac{sy(0) + \dot{y}(0) + cy(0)}{s + c}$$

$$= \frac{\dot{y}(0) + cy(0)}{c} = \frac{b}{c} + a$$

What did we miss? The condition $c > 0$.

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What did we miss? The condition $c > 0$.

# Roots and stability

Want to solve the differential equation

$$y^n + a_1 y^{n-1} + \ldots + a_{n-1} y' + a_n y = 0$$

Characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

If $a(\alpha) = 0$ then $y(t) = Ce^{\alpha t}$ is a solution to the differential equation

The general solution is

$$y(t) = \sum_k C_k(t) e^{\alpha_k t}$$

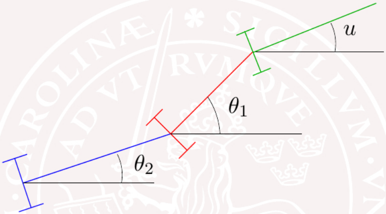
where $C_k(t)$ is a polynomial of degree $m - 1$ if $\alpha_k$ is a root of mult. $m$

$y(t) \to 0$ if all roots are in the open left half plane

# Eigenvalues - stability

$$G(s) = C(sI - A)^{-1}B = \frac{1}{\det(sI - A)} C\operatorname{adj}(sI - A)B$$

Eigenvalues: $\det(sI - A) = 0$.




$$\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} = v \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + v \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

How do the eigenvalues depend on speed $v$?

# Frequency analysis

- Frequency curves
  $u(t) = \sin \omega t$, $y(t) = A(\omega) \sin(\omega t + \varphi(\omega))$
  $A(\omega) = |G(i\omega)|$, $\varphi(\omega) = \arg G(i\omega)$
- Representation of $G(s)$ and $G(i\omega)$
- Nyquist diagram - complex number $G(i\omega)$
- Bode diagram – $|G(i\omega)|$ and $\arg G(i\omega)$
  $G = G_1 G_2 G_3 G_4 \ldots$

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