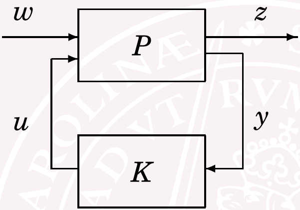
Lecture 6

- The H_∞ Optimization Problem
- Linear Quadratic Games
- Algebraic Riccati Equations
- State Space Solution to H_∞ Optimization
- LMIs for H_2 and H_∞ Optimization
- Risk Sensitive Control (LEQG)

The H_∞ Optimization Problem



$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$T_{zw} = \mathcal{F}_l(P, K)$$

Optimal control:

$$\min_{K-\text{stab}} \|T_{zw}\|_\infty$$

Suboptimal control: Given γ find an internally stabilizing controller K such that

$$\|T_{zw}\|_\infty < \gamma.$$

The optimal control problem is solved by iterating γ in the suboptimal problem.

H_∞ Optimization in Frequency Domain

A good exposition can be found in the book [Francis, 1987].

The *Youla parameterization* of all internally stabilizing controllers gives an affine dependence of T_{zw} on the Youla parameter $Q \in RH_\infty$

$$T_{zw} = T_1 - T_2 Q T_3, \quad T_k \in RH_\infty$$

Thus the H_∞ optimization problem becomes

$$\min_{Q \in RH_\infty} \|T_1 - T_2 Q T_3\|_\infty$$

The optimization in Q is convex, but infinite-dimensional

H_∞ Optimization in Frequency Domain

In a special case, the H_∞ optimization problem is equivalent to

$$\min_{F \in RH_\infty} \|R - F\|_\infty = \text{dist}(R, RH_\infty)$$

where R is unstable.

This problem of approximating an L_∞ function by an H_∞ function is a classical problem from the beginning of the 20th century (Markov, Caratheodory, Fejer, Nevanlinna, Pick, Sarason and many others). Nehari solved it in 1957.

State Space Solution: Recall LQ Control

If P satisfies the Riccati equation $A^TP + PA + Q - PBB^TP = 0$, then every solution to $\dot{x} = Ax + Bu$ with $\lim_{t\to\infty} x(t) = 0$ satisfies

$$
\begin{aligned}
&\int_0^\infty [x^TQx + u^Tu]dt \\
&= \int_0^\infty |u + B^TPx|^2 dt - 2\int_0^\infty (Ax + Bu)^T Pxdt \\
&= \int_0^\infty |u + B^TPx|^2 dt - 2\int_0^\infty \dot{x}^T Pxdt \\
&= \int_0^\infty |u + B^TPx|^2 dt - \int_0^\infty \frac{d}{dt}[x^TPx]dt \\
&= \int_0^\infty |u + B^TPx|^2 dt + x(0)^T Px(0)
\end{aligned}
$$

with the minimizing control law $u = -B^TPx$.

A Linear Quadratic Game

If X satisfies the Algebraic Riccati Equation

$$A^T X + XA + Q - X(B_u B_u^T - B_w B_w^T/\gamma^2)X \quad = \quad 0$$

then $\dot{x} = Ax + B_u u + B_w w$ with $x(0) = 0$ gives

$$\begin{aligned}&\int_0^\infty [x^T Q x + u^T u - \gamma^2 w^T w] dt \\ &= \int_0^\infty |u + B_u^T X x|^2 dt - \gamma^2 \int_0^\infty |w - B_w^T X x|^2 dt\end{aligned}$$

This can be viewed as a dynamic game between the player u, who tries to minimize and w who tries to maximize.

The minimizing control law $u = -B_u^T X x$ gives

$$\int_0^\infty [x^T Q x + u^T u] dt \leq \gamma^2 \int_0^\infty w^T w dt$$

so the gain from w to $z = (Q^{1/2} x, u)$ is at most γ.

Algebraic Riccati Equations

$$A^*X + XA + XRX + Q = 0$$

where $R = R^*$, $Q = Q^*$.

- The ARE is as important for control design as the Lyapunov equation is for system analysis.
- There are many solutions $X = X^*$ to ARE, the stabilizing one (which makes $A + RX$ stable) is unique!
- The ARE is a state space tool, which corresponds to factorization in frequency domain (recall spectral factorization in LQ Control).

How do we solve it?

Hamiltonian Matrix

Consider the $2n \times 2n$ matrix

$$H = \begin{pmatrix} A & R \\ -Q & -A^* \end{pmatrix}.$$

Lemma: Eigenvalues of H are symmetric with respect to the imaginary axis.

Proof: Introduce $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Then $J^{-1}HJ = -H^*$, so λ is an eigenvalue of H if and only if $-\bar{\lambda}$ is.

In particular, if there are no purely imaginary eigenvalues then there are precisely n stable and n unstable eigenvalues of H.

Stable Invariant Subspace

Under assumption of no purely imaginary eigenvalues, let

$$T = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in R^{2n\times n}$$

be a basis of the stable n-dimensional invariant subspace. Equivalently $HT = T\Lambda$ for some stable matrix $\Lambda \in R^{n\times n}$.

Lemma: If $\det(X_1) \neq 0$ then $X = X_2X_1^{-1}$ is a stabilizing solution to the ARE $A^*X + XA + XRX + Q = 0$

Proof: We are to prove

1) $X = X^*$.

2) X satisfies the ARE.

3) $A + RX$ is stable.

1) $HT = T\Lambda \Rightarrow T^*JHT = T^*JT\Lambda$. The matrix JH is symmetric then

$$T^*JT\Lambda = \Lambda^*T^*J^*T \quad \Leftrightarrow \quad T^*JT\Lambda + \Lambda^*T^*JT = 0.$$

So T^*JT satisfies the Lyapunov equation and Λ is stable. Hence $T^*JT = 0$, that is

$$X_2^*X_1 - X_1^*X_2 = 0 \quad \Leftrightarrow \quad X^* - X = 0.$$

2) & 3) Simple calculation gives

$$\begin{aligned} AX_1 + RX_2 &= X_1\Lambda, \\ -QX_1 - A^*X_2 &= X_2\Lambda. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} A + RX &= X_1\Lambda X_1^{-1} \\ -Q - A^*X &= X_2\Lambda X_1^{-1}. \end{aligned}$$

Thus $A + RX$ is stable and

$$XA + XRX = X_2\Lambda X_1 = -Q - A^*X$$

which implies the ARE.

How to solve the ARE

Under conditions

(H1) There are no pure imaginary eigenvalues of H.

(H2) $\det(X_1) \neq 0$ for some basis of stable invariant subspace.

we can find a stabilizing solution to ARE as follows:

1. Find a basis T for the stable invariant subspace, for example by Schur decomposition. If **(H1)** holds, then it has the dimension n.
2. Partition T as
$$T = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$
(H2) holds for some basis iff it holds for all basis.
3. Build $X = X_2 X_1^{-1}$.

Notation

$H \in \mathrm{dom}(\mathrm{Ric})$ if (H1) and (H2) hold for H.

$X = \mathrm{Ric}(H)$ is the stabilizing solution to ARE.

ARE for H_∞ Optimal State Feedback

Theorem: Consider $\dot{x} = Ax + B_u u + B_w w$, $x(0) = 0$, where (A, B_u) and (A, B_w) be stabilizable. Introduce the Hamiltonian

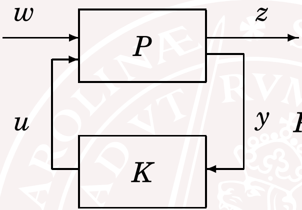
$$H_0 = \begin{pmatrix} A & B_w B_w^T/\gamma^2 - B_u B_u^T \\ -Q & -A^T \end{pmatrix}.$$

Then, the following conditions are equivalent:

1. There exists a stabilizing control law with $\int_0^\infty (x^T Q x + |u|^2) dt \le \gamma^2 \int_0^\infty |w|^2 dt$
2. H_0 has no purely imaginary eigenvalues.
3. $H_0 \in \mathrm{dom(Ric)}$.

Proof: The implication (3) $\Rightarrow$ (1) was proved on slide "A Linear Quadratic Game". For (2) $\Leftrightarrow$ (3), see [Zhou, p. 237].

Output Feedback Assumptions



$$P = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

(A1) (A, B_w, C_z) is stabilizable and detectable,

(A2) (A, B_u, C_y) is stabilizable and detectable,

(A3) $D_{zu}^* \begin{pmatrix} C_z & D_{zu} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$,

(A4) $\begin{pmatrix} B_w \\ D_{yw} \end{pmatrix} D_{yw}^* = \begin{pmatrix} 0 \\ I \end{pmatrix}$.

State Space H_∞ optimization

The solution involves two AREs with Hamiltonian matrices

$$
\begin{aligned}
H_\infty &= \begin{pmatrix} A & \gamma^{-2}B_wB_w^* - B_uB_u^* \\ -C_z^*C_z & -A^* \end{pmatrix} \\
J_\infty &= \begin{pmatrix} A^* & \gamma^{-2}C_z^*C_z - C_y^*C_y \\ -B_wB_w^* & -A \end{pmatrix}
\end{aligned}
$$

Theorem: There exists a stabilizing controller K such that $\|T_{zw}\|_\infty < \gamma$ if and only if the following three conditions hold:

1. $H_\infty \in \mathrm{dom}(\mathrm{Ric})$ and $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$,
2. $J_\infty \in \mathrm{dom}(\mathrm{Ric})$ and $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$,
3. $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, one such controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty = A + \gamma^{-2} B_w B_w^* X_\infty + B_u F_\infty + Z_\infty L_\infty C_y,$$

$$F_\infty = -B_u^* X_\infty,\ L_\infty = -Y_\infty C_y^*,$$

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

Furthermore, the set of all stabilizing controllers such that $\|T_{wz}\|_\infty < \gamma$ can be explicitly obtained as lower LFT (see [Zhou,p. 271]).

[Doyle J., Glover K., Khargonekar P., Francis B., *State Space Solution to Standard H^2 and H^∞ Control Problems*, IEEE Trans. on AC **34** (1989) 831–847.]

Idea of Proof

The dynamic game viewpoint gives a solution in the case of full information, where both state and disturbance are measurable. This gives the first ARE.

This can be combined with a "worst case observer", finding the smallest disturbance compatible with available measurements. This gives the second ARE.

Combining the full information solution with the worst case observer, solves the dynamc game problem with limited measurement information, provided that the spectral radius condition holds.

What have we learned so far?

- H_∞ optimization is fundamental problem for robust synthesis.
- A dynamic game between controller and disturbance
- The state space approach gives easily implementable conditions and formulas.
- The Algebraic Riccati Equation is the main computational tool.

Enter LMIs !

Control Synthesis using LMIs

Dullerud+Paganini, A course in Robust Control Theory, Ch7.

- LMI for H_2 optimal state feedback
- The KYP lemma
- LMI for H_∞ optimal state feedback
- Matrix elimination lemma
- LMI for H_∞ optimal state feedback

Stationary Stochastic Processes

Fact: If A is Hurwitz and w is white noise with intensity I then the stationary solution to $\dot{x} = Ax + Bw$ has covariance $Exx^T = X$ satisfying

$$AX + XA^T + BB^T = 0$$

(A more correct way of writing the stochastic differential equation is $dx = Axdt + Bdv$)

H_2 optimal state feedback

Problem: Given $\dot{x} = Ax + B_w w + B_u u$ find a stabilizing control law $u = Kx$ that mininizes $E(|x|^2 + |u|^2)$.

Solution: Closed loop system is $\dot{x} = (A + B_u K)x + B_w w$, so $X = Exx^T$ satisfies

$$(A + B_u K)X + X(A + B_u K)^T + B_w B_w^T = 0$$

This is a linear constraint on (X, Y) where $Y := KX$.

Hence we have the convex problem:

$\text{Minimize } \text{trace}(X) + \text{trace}(YX^{-1}Y^T)$
subject to $X > 0$ and $AX + B_u Y + (AX + B_u Y)^T + B_w B_w^T = 0$.

The KYP lemma

Given A, B and $M = M^T$ where A has no eigenvalues on imag. axis.

The following are equivalent

$$(i) \qquad \begin{bmatrix} (i\omega I - A))^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (i\omega I - A))^{-1}B \\ I \end{bmatrix} < 0, \qquad \forall \omega$$

(ii) There exists $P = P^T$ such that

$$M + \begin{bmatrix} A^T P + PA & PB \\ B^* P & 0 \end{bmatrix} < 0$$

The KYP is a classical result connecting frequency domain to time domain.

Proof is nontrivial, and we will skip it here.

Final result: LMIs for The Output Feedback Case

A controller that gives $\|M_{cl}\|_\infty < 1$ exists iff there exist symmetric matrices $X > 0$, $Y > 0$ with $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$ such that

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_1 & C_1^T \\ B_1^T X & -I & D_{11}^T \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_1^T & B_1 \\ C_1 Y & -I & D_{11} \\ B_1^T & D_{11}^T & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0$$

where N_o, N_c are full rank matrices with

$$\mathrm{Im} N_o = \mathrm{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix},$$
$$\mathrm{Im} N_c = \mathrm{Ker} \begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$$

Comparison to Riccati Approach

- More expensive to solve LMIs than AREs
- Fewer assumptions
- Can introduce sparsenenss constraints (conservative solution)
- H_2 and H_∞ specifications can be merged

Risk-sensitive Optimal Control (LEQG)

Peter Whittle, Risk-sensitive Optimal Control (1990)

The risk-sensitive optimal controller includes the H_2 and H_∞ control problem as special cases.

Instead of minimizing $J = E(\mathcal{C})$ where

$$\mathcal{C} = \sum_0^T \begin{bmatrix} x \\ u \end{bmatrix} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{21} & Q_2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

we choose to minimize

$$J_\theta = -\frac{2}{\theta} \log \left(E \left(\exp \left(-\theta \mathcal{C}/2 \right) \right) \right)$$

Risk-sensitive Optimal Control

A Taylor-expansion gives

$$J_\theta = E(\mathcal{C}) - \frac{\theta}{4}\mathrm{var}(\mathcal{C}) + O(\theta^2)$$

For $\theta = 0$ we obtain the risk-neutral H_2 (LQG) case. Variations of the random cost $\mathcal{C}$ is seen as advantageous in the case $\theta > 0$ (risk-seeking) and disadvantageous if $\theta < 0$ (risk-adverse).

θ gives some freedom how to judge variance of expected cost.

There is a negative value θ_c so that when $\theta < \theta_c$, the cost J_θ will be infinite.

In fact, one can prove that

$$\theta_c = -\gamma_o^2$$

where γ_o is the optimal H_∞ norm.

Risk-sensitive Optimal Control

Assume

$$\begin{aligned}\mathcal{A}x + \mathcal{B}u &= \epsilon \\ y + \mathcal{C}x &= \eta\end{aligned}$$

Then ML-estimation means that we find sequence of (ϵ, η) minimizing $\mathcal{D} := \sum d_t$ where

$$d_t := \begin{bmatrix}\epsilon \\ \eta\end{bmatrix} \begin{bmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{bmatrix}^{-1} \begin{bmatrix}\epsilon \\ \eta\end{bmatrix}$$

(Note duality to $\mathcal{C}$).

Risk-sensitive Optimal Control

The risk-sensitive optimal control can be found by dynamic programming of the so called total stress

$$\mathcal{S} := \mathcal{C} + \theta^{-1}\mathcal{D}$$

One splits the optimizing into extremizing ”past stress up to time t” and ”future stress after time t”.

Extremizing over sequences $x_0, \ldots, x_T$ (max), and future u_t(min) and y_t(max) using Lagrange-multipliers λ, μ give nice stationary conditions.

Risk-sensitive Optimal Control

Stationary conditions at time t can be written

$$\begin{bmatrix} Q_1 & Q_{12} & \mathcal{A}^* \\ Q_{21} & Q_2 & \mathcal{B}^* \\ \mathcal{A} & \mathcal{B} & -\theta R_1 \end{bmatrix} \begin{bmatrix} x \\ u \\ \lambda \end{bmatrix}_\tau = \begin{bmatrix} 0 \\ 0 \\ \theta R_{12}\mu \end{bmatrix}_\tau, \qquad \tau \geq t$$

$$\begin{bmatrix} R_1 & R_{12} & \mathcal{A} \\ R_{21} & R_2 & \mathcal{C} \\ \mathcal{A}^* & \mathcal{C}^* & -\theta Q_1 \end{bmatrix} \begin{bmatrix} -\theta\lambda \\ -\theta\mu \\ x \end{bmatrix}_\tau = \begin{bmatrix} -\mathcal{B}u \\ -y \\ \theta Q_{12}u \end{bmatrix}_\tau, \qquad \tau \leq t$$

One has that

$$\theta \begin{bmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \epsilon \\ \eta \end{bmatrix}$$

For more details, see Whittle's book.