Discrete time mixed H_2/H_∞ control

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Introduction

Continuous time mixed H_2/H_∞ control problem:

- Zhou, Kemin, et al. "Mixed H_2 and H_∞ performance objectives. I. Robust performance analysis." Automatic Control, IEEE Transactions on 39.8 (1994): 1564-1574.
- Doyle, John, et al. "Mixed H_2 and H_∞ performance objectives. II. Optimal control." Automatic Control, IEEE Transactions on 39.8 (1994): 1575-1587.
- ...

Discrete time mixed H_2/H_∞ control problem:

- Muradore, Riccardo, and Giorgio Picci. "Mixed H_2/H_∞ control: the discrete-time case." Systems & control letters 54.1 (2005): 1-13.

Some notations

- w is deterministic disturbance, w_0 is stochastic disturbance
- $\partial D \triangleq \{z : |z| = 1\}$
- For stochastic processes x and y, $x \perp y$ means $E(x - \bar{x})(y - \bar{y})^T = 0$, where $\bar{x} = Ex$, $\bar{y} = Ey$.

Mixed H_2/H_∞ control problem

Let $G(z)$ be a discrete-time system with realization

$$\begin{aligned} \sigma x &= Ax + B_0 w_0 + B_1 w + B_2 u \\ z_0 &= C_0 x + D_{00} w_0 + D_{01} w + D_{02} u \\ z &= C_1 x + D_{10} w_0 + D_{11} w + D_{12} u \\ y &= C_2 x + D_{20} w_0 + D_{21} w \end{aligned}$$

which satisfies the following assumptions:

A1. (A, B_2) is stabilizable and (C_2, A) is detectable,

A2. D_{02} has full column rank and D_{20} has full row rank,

A3. $\begin{bmatrix} A - zI & B_2 \\ C_0 & D_{02} \end{bmatrix}$ has full column rank $\forall z \in \partial D$,

A4. $\begin{bmatrix} A - zI & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank $\forall z \in \partial D$,

A5. $w \in P$ is a bounded power signal and $w_0 \sim GW(0, I)$ is a normalized Gaussian white noise. Moreover, $w_0(k) \perp w(j)$, $\forall k \geq j$,

A6. the initial condition $x(0) \sim G(x_0, R_0)$ is independent of w_0, $\forall k \geq 0$.

Mixed H_2/H_∞ control

The mixed H_2/H_∞ problem: find u^* and w^* such that

$$J_0(w^*, u^*) \geq J_0(w, u^*), \;\; J_2(w^*, u^*) \leq J_2(w^*, u)$$

is solvable, if and only if, the coupled algebraic Riccati equations

$$\begin{aligned}
&(A + B_2 F_2)^T X_\infty \left(I - \gamma^{-2} B_1 B_1^T X_\infty \right)^{-1} (A + B_2 F_2) \\
&- X_\infty + (C_1 + D_{12} F_2)^T (C_1 + D_{12} F_2) = 0
\end{aligned}$$

$$\begin{aligned}
&\left(A_C + \left(B_1 - B_2 R_{02}^{-1} D_{02} D_{01} \right) H_\infty \right)^T X_2 \left(A_C \right. \\
&\quad \left. + \left(B_1 - B_2 R_{02}^{-1} D_{02} D_{01} \right) H_\infty \right) - X_2 \\
&\quad + (C_0 + D_{21} H_\infty)^T \tilde{R}_{02} (C_0 + D_{21} H_\infty) = 0
\end{aligned}$$

$$\begin{aligned}
&\left(A_F + \left(B_1 - B_1 D_{20}^T R_{20}^{-1} D_{21} \right) H_\infty \right) Y_2 \left(I \right. \\
&\left. + C_2^T R_{20}^{-1} C_2 Y_2 \right)^{-1} \left(A_F + \left(B_1 - B_1 D_{20}^T R_{20}^{-1} D_{21} \right) H_\infty \right)^T \\
&- Y_2 + B_0 \tilde{R}_{20} B_0^T = 0
\end{aligned}$$

Mixed H_2/H_∞ control

In the three Ricaati equations,

$$
\begin{aligned}
F_2 =& -\left(D_{02}^T D_{02} + B_2^T X_2 B_2\right)^{-1} \\
& \left(B_2^T X_2 \left(A + B_1 H_\infty\right) + D_{02}^T \left(C_0 + D_{01} H_\infty\right)\right) \\
L_2 =& -\left(\left(A + B_1 H_\infty\right) Y_2 \left(C_2 + D_{21} H_\infty\right)^T + B_0 D_{20}^T\right) \\
& \left(D_{20} D_{20}^T + \left(C_2 + D_{21} H_\infty\right) Y_2 \left(C_2 + D_{21} H_\infty\right)^T\right)^{-1} \\
A_C =& A - B_2 R_{02}^{-1} D_{02}^T C_0 \\
A_F =& A - B_0 D_{02}^T R_{20}^{-1} C_2 \\
R_{02} =& D_{02}^T D_{02} \\
\tilde{R}_{02} =& I - D_{02} R_{02}^{-1} D_{02} T \\
R_{20} =& D_{20} D_{20}^T \\
\tilde{R}_{20} =& I - D_{20}^T R_{20}^{-1} D_{20}
\end{aligned}
$$

Mixed H_2/H_∞ control

The three Ricaati equations have stabilizaing solutions $X_\infty \geq 0$, $X_2 \geq 0$ and $Y_2 \geq 0$.
The optimal controller $u^* = K(z)y$ has a realization given by

$$
\begin{aligned}
\sigma x_K &= (A + B_1 H_\infty + B_2 F_2 +_2 C_2 + L_2 D_{21} H_\infty) x_K - L_2 y \\
u^* &= F_2 x_K
\end{aligned}
$$

Moreover the optimal value of the performance functional $J_2^*(w^*, u^*, w_0)$ is

$$
J_2^* = \operatorname{tr}\left(B_0^T X_2 B_0 + D_{00}^T D_{00}\right) + \operatorname{tr}\left(\left(B_2^T X_2 B_2 + D_{02}^T D_{02}\right) F_2 Y_2 F_2^T\right)
$$

A numerical example

A discrete-time system

$$\begin{aligned} \sigma x &= Ax + B_0 w_0 + B_1 w + B_2 u = 1.05x + w_0 + w - 0.55u \\ z_0 &= C_0 x + D_{00} w_0 + D_{01} w + D_{02} u = 0.94x + 1.36u \\ z &= C_1 x + D_{10} w_0 + D_{11} w + D_{12} u = -0.54x + 0.57u \\ y &= C_2 x + D_{20} w_0 + D_{21} w = -0.59x + 0.52w_0 + w \end{aligned}$$

Use MATLAB to solve nonlinear equations. The optimal mixed H_2/H_∞ controller is

$$\begin{aligned} \sigma x_K &= (A + B_1 H_\infty + B_2 F_2 +_2 C_2 + L_2 D_{21} H_\infty) x_K - L_2 y \\ &= 0.9376 x_K + 0.1406 y \\ u^* &= F_2 x_K = 0.3457 x_K \end{aligned}$$

The optimal H_2 cost is $J_2^* = 3.42$.

Future work

- Compare the mixed H_2/H_∞ performance with pure H_2 and H_∞ controllers
- LMI, Nash game (MIMO)
- Minimize H_∞ cost with constraint of H_2 cost.