Where to Place the Poles?

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Where to Place the Poles?

Theme: Be aware where you place them!

Introduction

- A simple idea
- Strong impact on development of control theory
- The only constraint is reachability and observability
- The robustness debate

  Classic control vs State feedback
- Easy to apply for simple systems
- Polynomial equations notoriously badly conditioned $z^n = 0$
- How to choose closed loop poles - The Million $ question

  How do the closed loop poles influence performance
  How do the closed loop poles influence robustness
  A bit of history - Mats Lilja's PhD thesis TFRT 1031 (1989)
  Model reduction

First Order Systems

- State: variables required to characterize storage of mass, momentum and energy
- Many systems are approximately of first order
- The key is that the storage of mass, momentum and energy can be captured by one parameter
- Examples
  - Velocity of car on the road
  - Control of velocity of rotating system
  - Electric systems where energy storage is essentially in one capacitor or one inductor
  - Incompressible fluid flow in a pipe
  - Level control of a tank
  - Pressure control in gas tank
  - Temperature in a body with essentially uniform temperature distribution (e.g. steam filled vessel)

Second Order Systems

- Two states because storage of mass, momentum and energy can be captured by two parameter
- Examples
  - Position of car on the road
  - Control of angle of rotating system
  - Stabilization of satellites
  - Electric systems where energy is stored in two elements (inductors or capacitors)
  - Levels in two connected tanks
  - Pressure in two connected vessels

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2. Poleplacement Design
3. Examples and Design Rules
4. Model Reduction
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Pole Placement

Process and controller

$$d_p(s)Y(s) = n_p(s)U(s) \qquad d_c(s)U(s) = n_{cff}(s)R(s) - n_c(s)Y(s)$$

Closed loop transfer function

$$G_{yr}(s) = \frac{n_p(s)n_{cff}(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)} = \frac{n_p(s)n_{cff}(s)}{d_{cl}(s)}$$

Determine $d_c(s)$ and $n_c(s)$ to give the desired closed loop polynomial $d_{cl}(s)$. The zeros can be partially influenced through $n_{cff}(s)$.

Parameter count

- $\deg d_c + \deg n_c + 1 = \deg d_{cl}$
- Introduce unknown coefficients and solve linear equation

The Diophantine Equation

The equation

$$3x + 2y = 1,$$

where x and y are integers has the solution is $x = 1$ and $y = -1$. Many other solutions can be obtained by adding 2 to x and subtracting 3 from y.

The equation

$$6x + 4y = 1,$$

cannot have a solution, because the left hand side is even and the right hand side is odd.

The equation

$$6x + 4y = 2,$$

has a solution, because we can divide by 2 and obtain the first equation.

Main Result

Let a, b, c, x and y be integers, the equation

$$ax + by = c$$

has a solution if and only if the greatest common factor of a and b divides c. If the equation has a solution x_0 and y_0 then $x = x_0 - bn$ and $y = y_0 + an$, where n is an arbitrary integer, is also a solution.

- Integers and polynomials same algebra, add, subtract, divide with remainder (size replaced by degree)
- Euclid's algorithm holds for polynomials (the same algebra add & mult!)

Proof - Euclid's Algorithm

Assume that the degree of a is greater than or equal to the degree of b. Let $a^0 = a$ and $b^0 = b$. Iterate the equations

$$a^{n+1} = b^n \qquad b^{n+1} = a^n \mod b^n$$

until $b^{n+1} = 0$. The greatest common divisor is then $g = b^n$. If a and b are co-prime we have $b^n = 1$. Back-tracking we find that

$$ax + by = b^n = g$$

where the polynomials x and y can be found by keeping track of the quotients and the remainders in the iterations. When a and b are co-prime ($g = 1$) we get

$$ax + by = 1$$

Multiplying x and y by c gives the original equation $ax + by = c$. When a and b have a common factor the largest common divisor of a and b must be a factor of c.

An Algorithm

Let g be the greatest common divisor of a and b and let u and v be the minimal degree solutions to $ax + by = 0$.

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} = \begin{pmatrix} g & x & y \\ 0 & u & v \end{pmatrix}$$

Make row transformations to transform (Gaussian elimination)

$$A^{(0)} = \begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \quad \text{to} \quad A^{(n)} = \begin{pmatrix} g & x & y \\ 0 & u & v \end{pmatrix}$$

It follows from Euclid's algorithm that $g = A_{11}^{(n)}$ is the largest common divisor of a and b, and that a and b are co-prime if and only if $A_{11}^{(n)} = 1$. The equation

$$ax + by = c$$

has a solution if $A_{11}^{(n)}$ is a factor of c.

Non-uniqueness I

Closed loop characteristic polynomial

$$d_p d_c + n_p n_c = d_{cl}, \qquad C = n_c/d_c$$

If d_{c0}, n_{c0} is a solution then $d_{c0} - qn_p, n_{c0} + qd_p$, where q is an arbitrary polynomial. Many different choices

- Minimal numerator degree $\deg n_c < \deg d_p$, generically
  $\deg d_c = \deg n_c = \deg d_p - 1,\ \deg d_{cl} = 2\deg d_p - 1$
  (Luenberg)
  $\deg d_c = \deg d_p,\ d_p d_c + n_p n_c = d_{cl}, \quad \deg n_c = \deg d_p - 1, \deg d_{cl} = 2\deg d_p$ (Kalman)
- Minimal denominator degree $\deg d_c \leq \deg n_p$ (controller has excess of zeros, derivative action). Generically
  $\deg d_c = \deg n_p - 1,\ \deg n_c = \deg d_p - 1,\ \deg d_{cl} = \deg d_p + \deg n_p - 1$
  $\deg n_p = 0,\ d_c = 1,\ \deg n_c = \deg d_p - 1,\ \deg d_{cl} = \deg d_p$
- Integral action: Add s as an extra factor of $d_p(s)$ solve for d_c and n_c and the controller is then $C = d_c(s)/(sn_c(s))$.

Non-uniqueness II

Process and controller transfer functions

$$P(s) = \frac{n_p(s)}{d_p(s)} \qquad C(s) = \frac{n_c(s)}{d_c(s)}$$

Closed loop characteristic equation

$$d_p(s)n_c(s) + n_p(s)d_c(s) = d_{cl}(s)$$

If $C_0 = n_{c0}(s)/d_{c0}(s)$ is a controller that gives the closed loop characteristic polynomial $d_{cl}(s)$ then the controller

$$C(s) = \frac{n_{c0}(s) + q(s)d_p(s)}{d_{c0}(s) - q(s)n_p(s)}$$

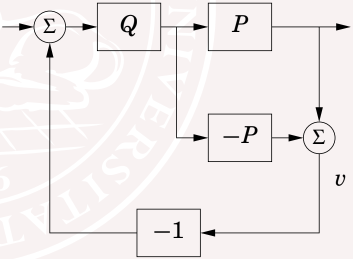
where $q(s)$ is an arbitrary polynomial also gives char. pol $d_{cl}(s)$.

$$d_p(s)\big(d_{c0}0(s) - q(s)n_p(s)\big) + n_p(s)\big(n_{c0} + q(s)d_p(s)\big) =$$
$$d_p(s)n_c(s) + n_p(s)d_c(s) = d_{cl}(s)$$

Youla-Kucera Parametrization 1

Consider a process with a *stable* transfer function P. Let the desired transfer functionfrom reference to output be T . The requirement can be realized by feedforward with the transfer function Q, where $T = PQ$. Since Q must be stable T and P should have the same zeros in the right half plane. The transfer function T can also be obtained by error feedback with the controller

$$C = \frac{Q}{1 - PQ}$$



Q arbitrary stable rational trf. GoF:

$$T = PQ \quad S = 1 - T = 1 - PQ$$
$$PS = P(1 - PQ) \quad CS = Q$$

All stabilizing controllers can be represented by C for some Q.

Youla-Kučera Parametrization 2

Process transfer function $P = B/A$, where A and B are stable co-prime rational functions. Controller function $C_0 = G_0/F_0$ stabilizes P. All stabilizing controllers are given by

$$C = \frac{G_0 + QA}{F_0 - QB}$$

Q is an arbitrary stable rational transfer function. GoF:

$$T = \frac{B(G_0 + QA)}{AF_0 + BG_0} \qquad PS = \frac{B(F_0 - QB)}{AF_0 + BG_0}$$

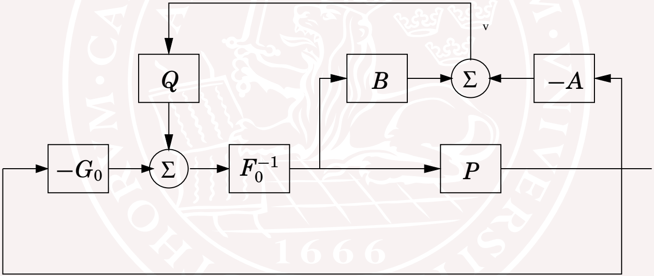
$$CS = \frac{A(G_0 + QA)}{AF_0 + BG_0} \qquad S = \frac{A(F_0 - QB)}{AF_0 + BG_0}$$

The system is stable since the rational function $AF_0 + BG_0$ has all its zeros in the left half plane and A, B, F_0, G_0 and Q are stable rational functions.

Block Diagram Interpretation

Controller

$$C = \frac{G_0 + QA}{F_0 - QB} \qquad F_0 U = -G_0 Y + Q(BU - AY)$$



Notice that the input to Q is nominally zero

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Robustness

- Intuitively we may expect that well-damped closed loop poles would guarantee robustness
- Unfortunately this is not true!!!
- Always necessary to check robustness if it is not part of design process
- Always check requirements that are not explicit requirements in the design procedure, particularly if you optimize
- Looking at the Gang of Four is a good idea
- A long forgotten problem (Mats Lilja's PhD #31 1989)
- Two examples
- Two simple design rules

Example 1

Consider a first order system with PI control

$$P(s) = \frac{b}{s+a} = \frac{1}{s+1}, \quad C(s) = k + \frac{k_i}{s}$$

where the controller parameters are chosen to give a closed loop system with the characteristic polynomial $s^2 + \omega_0 s + \omega_0^2$.

Characteristic polynomial

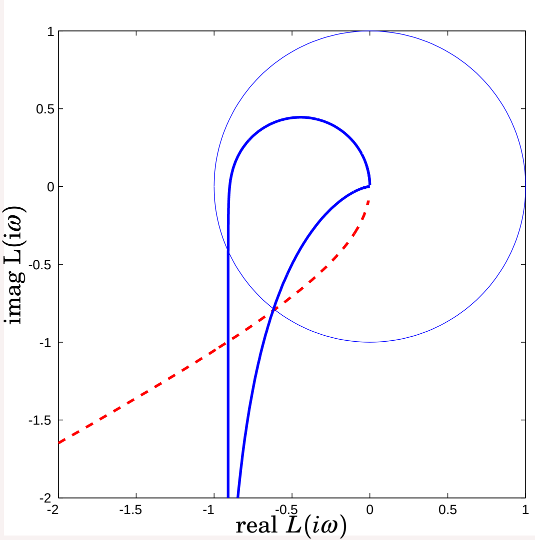
$$s(s+a) + b(k_p s + k_i) = s^2 + \omega_0 s + \omega_0^2$$

Controller parameters

$$k_p = \frac{\omega_0 - a}{b} = \omega_0 - 1, \qquad k_i = \frac{\omega_0^2}{b} = \omega_0^2$$

What is special about $\omega_0 = 1$? What does it mean that k_p is negative?

Nyquist Plot $\omega_0/a = 0.1$, 1and 10 (red)



The Gang of Four

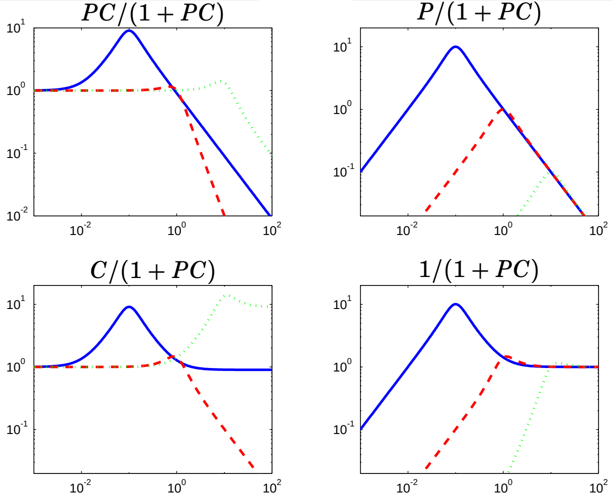
The Gang of Four is given by

$$\frac{PC}{1+PC} = \frac{(\omega_0 - a)s + \omega_0^2}{s^2 + \omega_0 s + \omega_0^2} \qquad \frac{P}{1+PC} = \frac{bs}{s^2 + \omega_0 s + \omega_0^2}$$

$$\frac{C}{1+PC} = \frac{((\omega_0 - a)s + \omega_0^2)(s+a)}{b(s^2 + \omega_0 s + \omega_0^2)} \qquad \frac{1}{1+PC} = \frac{s(s+a)}{s^2 + \omega_0 s + \omega_0^2}$$

We will investigate the properties of the Gang of Four for $\omega_0/a = 0.1$, 1 and 10.

Gain Curves for the Gang of Four



Looks OK for $\omega_0/a = 1$ and 10 BUT not for $\omega_0 = 0.1$ (blue curves)

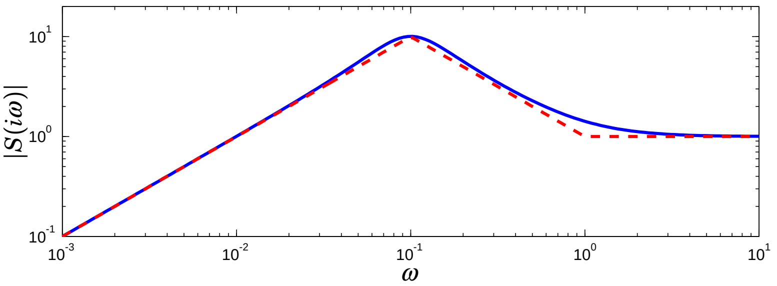
Comments

- We have made what looks like a perfectly reasonable pole placement design with closed loop poles having reasonable damping: $\zeta_0 = 0.5$.
- The results look good for $\omega_0/a = 1$ and 10
- The design for $\omega_0/a = 0.1$ have very high sensitivities $M_s = 9.4$ and $M_t = 10$
- It is apparently important where we place the poles
- Can we understand what goes on and fix it?

The Sensitivity Function

We have for $a = 1$ and $\omega_0 = 0.1$, $M_s \approx \dfrac{0.1}{0.011} = 9$ (9.4)

$$S = \frac{(s+a)s}{s^2 + \omega_0 s + \omega_0^2} = \frac{(s+1)s}{s^2 + 0.1s + 0.01} = \frac{d_p(s)d_c(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)}$$



What creates the peak?

Model Reduction

For small ω_0 we can approximate the process model instead of canceling the fast process pole

$$P(s) = \frac{b}{s+a} \approx \frac{b}{a}$$

Use an I controller

$$C(s) = \frac{k_i}{s}$$

Closed loop characteristic polynomial with true model

$$s(s+a) + bk_i$$

Sketch root loci in both cases!

Generalization

Transfer functions of process and controller

$$P(s) = \frac{n_p(s)}{d_p(s)}, \qquad C(s) = \frac{n_c(s)}{d_c(s)},$$

Sensitivity functions

$$S(s) = \frac{1}{1+PC} = \frac{d_p(s)d_c(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)}$$

At high frequencies we have $S \approx 1$. As the frequency decreases there will be a break-point at the process poles (zeros of d_p. To avoid having high sensitivities high frequency process poles must be matched by corresponding closed loop poles. In the example there was a process pole at $s = 1$ but the closed loop poles were at 0.1.

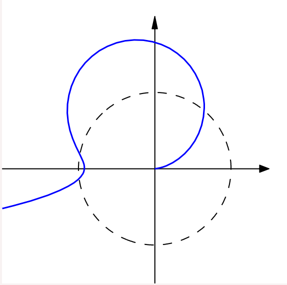
Example 2

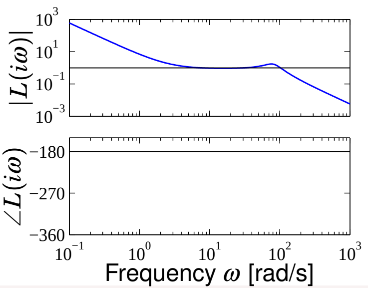
Consider the process

$$P(s) = \frac{0.5s + 1}{s^2}$$

$\omega_c = 10$, $\zeta_c = 0.707$, $\omega_o = 20$ and $\zeta_o = 0.707$ we get

$$C(s) = \frac{-11516s + 40000}{s^2 + 42.4s + 6657.9}.$$





Example 2

Consider the process

$$P(s) = \frac{b_1 s + b_2}{s^2}$$

Desired closed loop characteristic polynomial

$$c(s) = (s^2 + 2\zeta_c\omega_c s + \omega_c^2)(s^2 + 2\zeta_o\omega_o s + \omega_o^2)$$

We have

$$s^2(s^2 + f_1 s + f_2) + (b_1 + b_2)(g_0 s + g_1) = c(s)$$

Identification of coefficients of equal powers of s gives

$$f_1 = 2(\zeta_o\omega_o + \zeta_c\omega_c)$$

$$f_2 = \frac{\omega_o^2 + \omega_c^2 + 4z_o z_c\omega_o\omega_c - 2b_1(\zeta_0 w_c + \zeta_c w_o)w_o w_c + b_1^2 w_0^2 w_c^2}{b_2}$$

$$g_1 = \frac{2b_2(\zeta_o\omega_c + \zeta_c\omega_o)\omega_o\omega_c - b_1\omega_0^2\omega_c^2}{b_2^2}$$

$$g_2 = \omega_0^2\omega_c^2/b_2$$

Generalization

Transfer functions of process and controller

$$P(s) = \frac{n_p(s)}{d_p(s)}, \qquad C(s) = \frac{n_c(s)}{d_c(s)},$$

Sensitivity functions

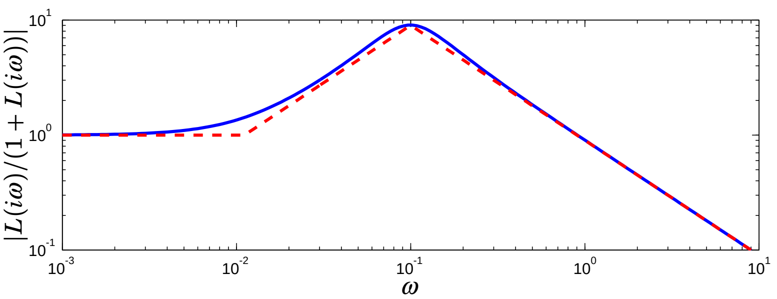
$$T(s) = \frac{PC}{1+PC} = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)}$$

At low frequencies we have $T \approx 1$. As the frequency increases there will be breakpoints at the process zeros of (zeros of n_p). To avoid having high sensitivities low frequency process zeros must be matched by corresponding closed loop poles. In the example there was a process zero at $s = 0.5$ but the slowest closed loop poles were at $s = 10$, hence a peak of ≈ 10.

Complementary Sensitivity

We have for $a = 1$ and $\omega_0 = 0.1$

$$T = \frac{(\omega_0 - 1)s + \omega_0^2}{s^2 + \omega_0 s + \omega_0^2} = \frac{-0.9s + 0.01}{s^2 + 0.1s + 0.01}$$



We have approximately $M_t \approx \dfrac{0.1}{0.01} = 10$ (10.04)

Design Rules for Poleplacement

- Formally only reachability and observability required
- To obtain robust closed loop systems the poles and zeros of the process must be taken into accound. Design rule.
- Choose bandwidth ω_b or dominating closed loop poles: classify poles and zeros as slow $< \omega_b$ or fast $> \omega_b$
- Slow unstable zeros (time delays) and fast unstable poles restrict the choice of closed loop bandwidth ω_b
- Design rule: *Pick closed loop poles close to slow stable process zeros and fast stable process poles.* Picking closed loop poles and zeros identical to slow stable zeros and fast stable poles give cancellations and simple calculations.
- Violating the design rule leads to closed loop systems that are not robust. Åström Murray Feedback Systems 365-366.

Unstable Poles and Zeros?

Unstable poles and zeros cannot be canceled therefore

- Bandwidth must be larger than the fastest unstable pole
- Bandwidth must be smaller than the slowest unstable zero

Summary of Limitations - Part 1

- A RHP zero z gives an upper bound to bandwidth

$$\frac{\omega_{gc}}{z} \leq \begin{cases} 0.5 & \text{for } M_s,\, M_t < 2 \\ 0.2 & \text{for } M_s,\, M_t < 1.4. \end{cases}$$

- A time delay T gives an upper bound to bandwidth

$$\omega_{gc} T \leq \begin{cases} 0.7 & \text{for } M_s,\, M_t < 2 \\ 0.4 & \text{for } M_s,\, M_t < 1.4. \end{cases}$$

- A RHP pole p gives a lower bound to bandwidth

$$\frac{\omega_{gc}}{p} \geq \begin{cases} 2 & \text{for } M_s,\, M_t < 2 \\ 5 & \text{for } M_s,\, M_t < 1.4. \end{cases}$$

Summary of Limitations - Part 2

- RHP poles and zeros must be sufficiently separated

$$\frac{z}{p} \geq \begin{cases} 7 & \text{for } M_s,\, M_t < 2 \\ 14 & \text{for } M_s,\, M_t < 1.4. \end{cases}$$

- RHP poles and zeros must be sufficiently separated

$$\frac{p}{z} \geq \begin{cases} 7 & \text{for } M_s,\, M_t < 2 \\ 14 & \text{for } M_s,\, M_t < 1.4 \end{cases}$$

- The product of a RHP pole and a time delay cannot be too large

$$pT \leq \begin{cases} 0.16 & \text{for } M_s,\, M_t < 2 \\ 0.05 & \text{for } M_s,\, M_t < 1.4. \end{cases}$$

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When are Two Systems Close?

- For stable systems

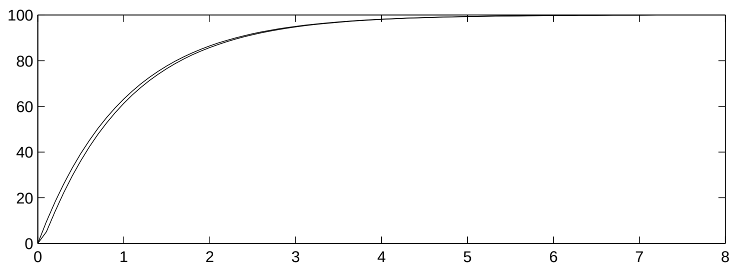
$$\delta(P_1, P_2) = \max_{\omega} |P_1(i\omega) - P_2(i\omega)|$$

  over relevant frequency ranges is a measure of of closeness of two stable processes.
- Is this a good measure?
- Are there other alternatives?
- A long story

    Gap metric (Zames)
    Graph metric coprime factorization (Vidyasagar) $G = N/D$
    Vinnicombe's metric

Similar Open Loop Different Closed Loop

$$P_1(s) = \frac{1000}{s+1}, \qquad P_2(s) = \frac{1000a^2}{(s+1)(s+a)^2}$$



Complementary sensitivity functions with unit feedback $C = 1$

$$T_1 = \frac{1000}{s+1001}, \qquad T_2 = \frac{10^7}{(s-287)(s^2+86s+34879)}$$

Different Open Loop Similar Closed Loop

The systems

$$P_1(s) = \frac{1000}{s+1}, \qquad P_2(s) = \frac{1000}{s-1}$$

are very different because P_1 is stable and P_2 unstable. The complementary sensitivity functions obtained with unit feedback are

$$T_1(s) = \frac{1000}{s+1001} \qquad T_2(s) = \frac{1000}{s+999}$$

These closed loop systems are undistinguishable

The Graph Metric

We know how to compare stable systems. What to do with unstable systems? Let

$$P(s) = \frac{B(s)}{A(s)}$$

where A and B are polynomials. Choose a stable polynomial C whose degree is not lower than the degrees of A and B, then

$$P(s) = \frac{\dfrac{B(s)}{C(s)}}{\dfrac{B(s)}{C(s)}} = \frac{N(s)}{D(s)}$$

Compare the numerator and denominator transfer functions jointly (the graph).

Many Ways to Choose D

Two rational functions D and N are called coprime if there exist rational functions X and Y which satisfy the equation

$$XD + YN = 1$$

The condition for coprimeness is essentially that $D(s)$ and $N(s)$ do not have any common factors.

Let $D^*(s) = D(-s)$. A factorization $P = N/D$ such that

$$DD^* + NN^* = 1$$

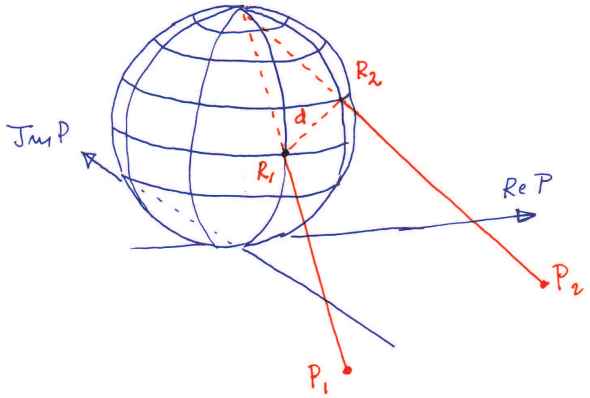
is called a coprime factorization of P.

Vinnicombe's nu-Gap Metric

If a winding number constraint is satisfied Vinnicombe's ν-gap metric can be defined as

$$\delta_\nu(P_1, P_2) = \sup_\omega \frac{|P_1(i\omega) - P_2(i\omega)|}{\sqrt{(1 + |P_1(i\omega)|^2)(1 + |P_2(i\omega)|^2)}}$$

Geometric Interpretation - The Riemann Sphere



Feedback Interpretation

Consider systems with the transfer functions P_1 and P_2. Compare the complementary sensitivity functions for the closed loop systems obtained with a controller C that stabilizes both systems.

$$\delta(P_1, P_2) = \left|\frac{P_1C}{1+P_1C} - \frac{P_2C}{1+P_2C}\right| = \left|\frac{(P_1-P_2)C}{(1+P_1C)(1+P_2C)}\right|$$

For frequencies where the maximum sensitivity is large we have

$$\delta(P_1, P_2) \approx M_{s1}M_{s2}|C(P_1-P_2)|$$

It can be shown that $\max_\omega \delta$ is a good measure of closeness of processes.

Vinnicombes nu-gap metric corresponds to $C = 1$ and maximization over frequencies, i.e. unit feedback.

The Winding Number Constraint

Consider two systems with the normalized coprime factorizations

$$P_1 = \frac{D_1}{N_1}, \qquad P_2 = \frac{D_2}{N_2}$$

To compare the systems it must be required that

$$\frac{1}{2\pi}\Delta \arg_\Gamma (N_1 N_2^* + D_1 D_2^*) = 0$$

where Γ is the Nyquist contour. In the polynomial representation this condition implies

$$\frac{1}{2\pi}\Delta \arg_\Gamma (B_1 B_2^* + A_1 A_2^*) = \deg A_2$$

The winding number constraint!

Robustness

Effect of small process changes on $T = PC/(1+PC)$

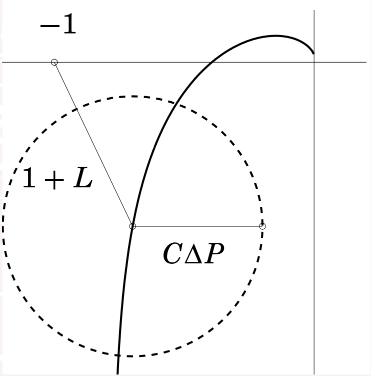
$$\frac{dT}{dP} = \frac{dP}{P} - \frac{CdP}{1+PC} = \frac{1}{1+PC}\frac{dP}{P} = S\frac{dP}{P}$$

How much can the process be changes without making the system unstable?

$$|C\Delta P| < |1+PC|$$

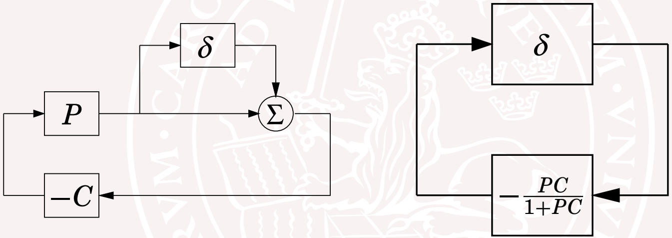
or

$$\frac{|\Delta P|}{|P|} < \frac{1}{|T|}$$



Another View of Robustness

A feedback system where the process has multiplicative uncertainty, i.e. $P(1+\delta)$, where δ is the relative error, can be represented with the following block diagrams



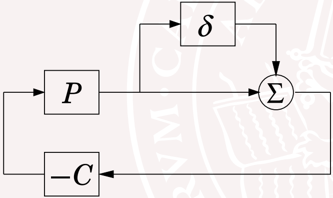
The small gain theorem gives the stability condition

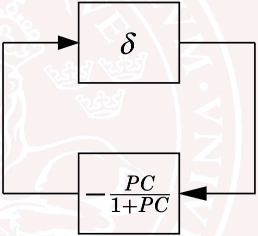
$$|\delta P| < \left|\frac{1+PC}{PC}\right| = \frac{1}{|T|}$$

Another View of Robustness

A feedback system where the process has multiplicative uncertainty, i.e. $P(1+\delta)$, where δ is the relative error, can be represented with the following block diagrams

v





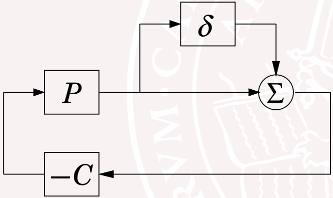
The small gain theorem gives the stability condition

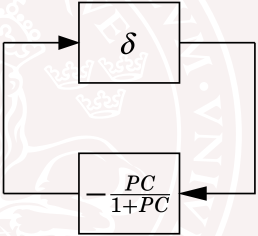
$$|\delta P| < \left|\frac{1+PC}{PC}\right| = \frac{1}{|T|}$$

Another View of Robustness

A feedback system where the process has multiplicative uncertainty, i.e. $P(1+\delta)$, where δ is the relative error, can be represented with the following block diagrams

v





The small gain theorem gives the stability condition

$$|\delta P| < \left|\frac{1+PC}{PC}\right| = \frac{1}{|T|}$$

Robustness

Additive perturbations $P \rightarrow P + \Delta P$, ΔP stable

$$\frac{|\Delta P(i\omega)|}{|P(i\omega)|} < \frac{|P(i\omega)C(i\omega)|}{|1 + P(i\omega)C(i\omega)|} = \frac{1}{|T(i\omega)|}$$

For normalized Co-prime factor perturbations
$P = N/D \;\rightarrow\; (N + \Delta N)(D + \Delta D)$ this generalizes to

$$||(\Delta N(i\omega), \Delta D(i\omega))|| < \frac{1}{\gamma(\omega)}$$

where (notice frequency by frequency comparison!)

$$\gamma = \bar{\sigma} \begin{pmatrix} \dfrac{1}{1 + P(i\omega)C(i\omega)} & \dfrac{P(i\omega)}{1 + P(i\omega)C(i\omega)} \\ \dfrac{P(i\omega)}{1 + P(i\omega)C(i\omega)} & \dfrac{P(i\omega)C(i\omega)}{1 + P(i\omega)C(i\omega)} \end{pmatrix} = \frac{\sqrt{(1 + |P(i\omega)|^2)(1 + |C(i\omega)|^2)}}{|1 + P(i\omega)C(i\omega)|}$$

The Generalized Stability Margin

$\mathcal{H}_\infty$ control

$$H(P,C) = \begin{pmatrix} \frac{PC}{1+PC} & \frac{P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{pmatrix} = \delta(P,-1/C)$$

$$\gamma = max_\omega \bar{\sigma}\big(H(i\omega)\big)$$

Stability margin

$$b = \frac{1}{\gamma}$$

Sensitivity to model errors (Vinnicombe). Design a controller C for the process P with stability margin b. If

$$\delta_\nu(P,P_1) < b$$

the controller is also stable for the process P_1

Where to Place the Poles?

1. Introduction
2. Poleplacement Design
3. Examples and Design Rules
4. Model Reduction
5. Summary

Theme: Be aware where you place them!

Summary

- Simple design method for SISO systems
- There are multivariable versions but they are complicated
- Probably quickest way to introduce design
- Robustness and design rules are important
- Useful insights Euclid's algorithm and Youla parametrization
- Polynomials are bad numerically matrix calculations much more robust
- Model reduction
- Understand when two systems are similar

Reading Suggestions

Åström Murray: Feedback Systems - An Introduction for Scientists and Engineers, Princeton 2008 (use Richards home page to download the book).

- Design rules for pole placement pp 365–366
- Vinnicombe ν-gap metric pp 349–352
- GoF Ch 11.
- Use index for other things.

Glenn Vinnicombe: Uncertainty and Feedback - $\mathcal{H}_\infty$ loop-shaping and the ν-gap metric. Imperial College Press 2001.