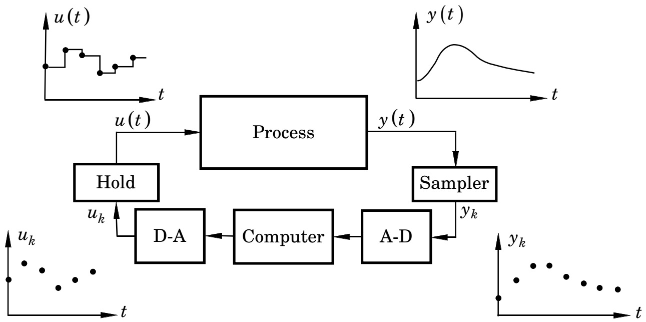
Lecture 1:

Advanced Digital Control

Goals of the course

- To better understand discrete-time systems
- To better understand computer-controlled systems



Contents of the course

- Sampling of systems and signals
- Analysis of discrete-time systems
  - I/O models
  - More about z-transform and δ-transform
- Hybrid systems
- Design methods
  - Reference values and integrators
  - I/O models and connection to state-space models
  - Optimal control, minimum variance and LQG

IT tools

Course information, available tools

- WWW at

  `http://www.control.lth.se/Education/`
  `DoctorateProgram/advanced-digital-control/`

Figures generated with macros available

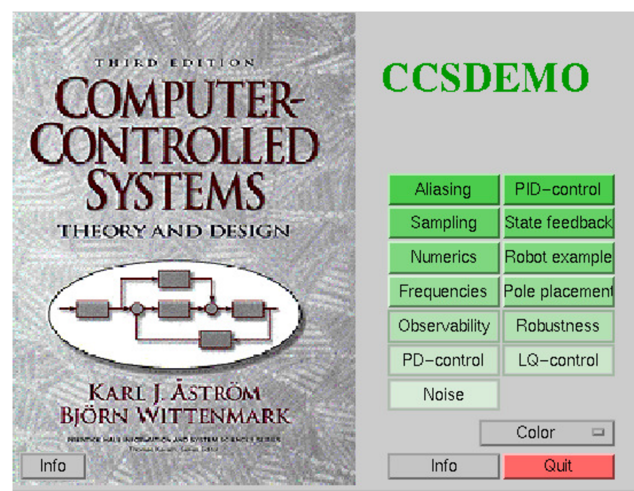
- Matlab and Simulink

  `/home/ccs/matlab/ccs5/chapterX`

- Interactive tool `ccsdemo`

CCSDEMO

Matlab and Simulink tool for the course

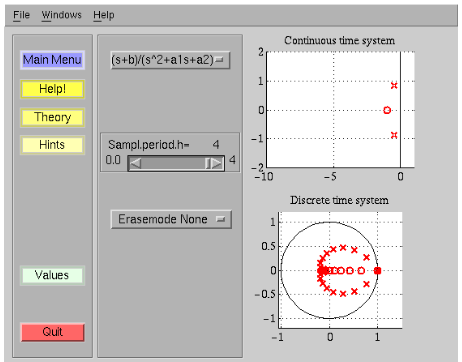


CCS — Example

Sampling of

$$\frac{s+b}{s^2+a_1 s+a_2}$$

for different sampling intervals

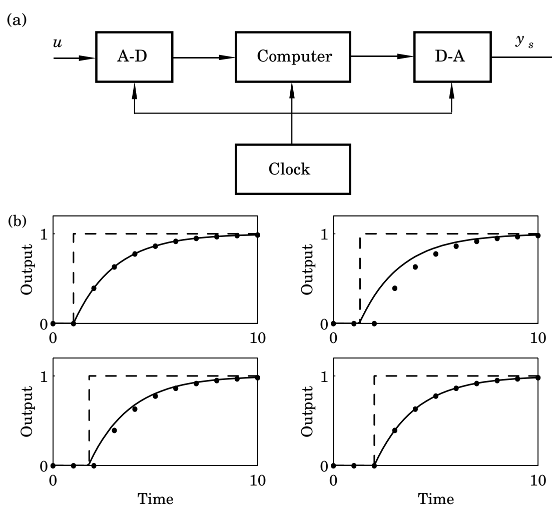


Sampling of signals and systems

- Introduction to computer-controlled systems
- Sampling of signals
- Sampling of systems
  - State-space models
  - Input-output models
- Discrete-time systems
  - Disregard intersample behavior

Problems with sampled-data systems 1

Time dependence



A naive approach

Control of the arm of a disk drive

$$G(s)=\frac{k}{Js^2}$$

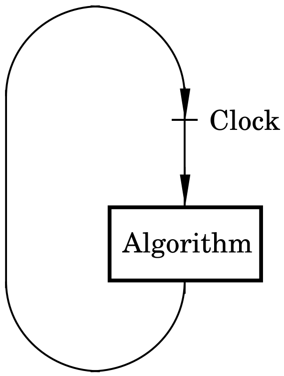
Continuous time controller

$$U(s)=\frac{bK}{a}U_c(s)-K\frac{s+b}{s+a}Y(s)$$

Discrete time controller

$$u(t_k)=K\left(\frac{b}{a}u_c(t_k)-y(t_k)+x(t_k)\right)$$
$$x(t_k+h)=x(t_k)+h\left((a-b)y(t_k)-ax(t_k)\right)$$

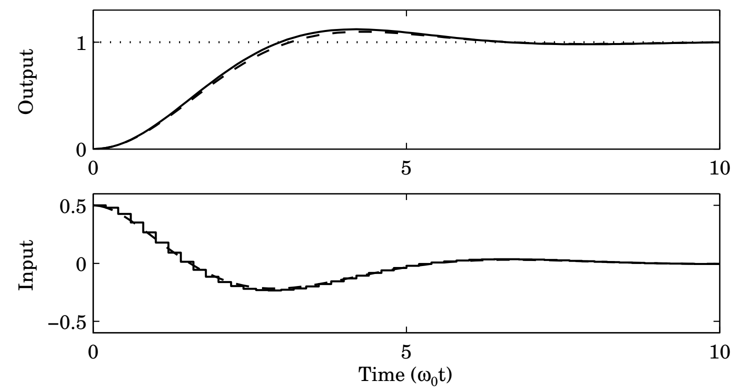
Control of the double integrator




```
y: = adin(in2) {read process value}
u:=K*(a/b*uc-y+x)
dout(u)  {output control signal}
newx:=x+h*((b-a)*y-b*x)
```

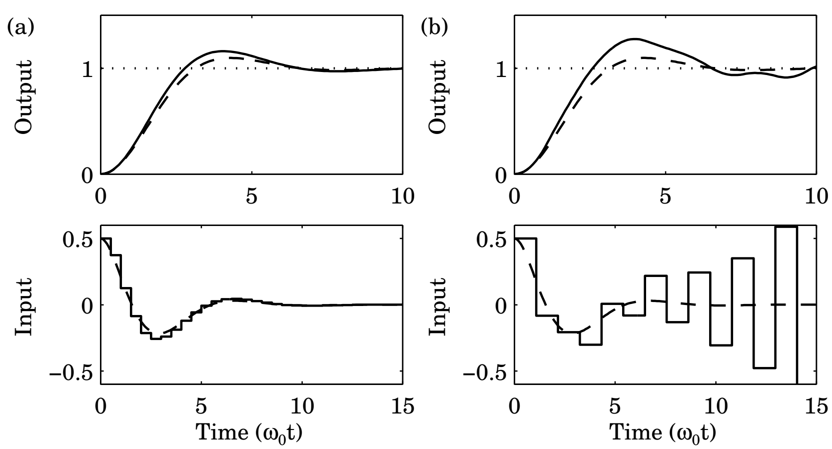
Control of the double integrator – Short sampling period

Sampling period $h = 0.2/\omega_0$



Control of the double integrator – Increased sampling period

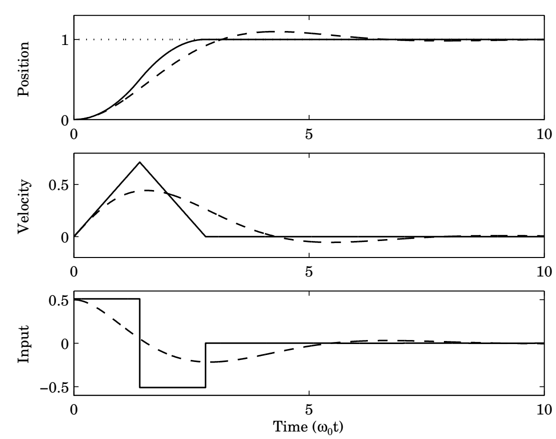
a) $h = 0.5/\omega_0$ b) $h = 1.08/\omega_0$



Better performance?

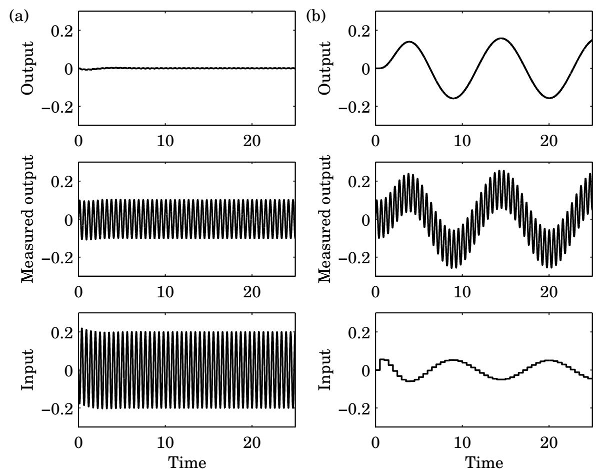
Dead-beat control $h = 1.4/\omega_0$

$$u(t_k) = t_0 u_c(t_k) + t_1 u_c(t_{k-1}) - s_0 y(t_k) - s_1 y(t_{k-1}) - r_1 u(t_{k-1})$$



Sinusoidal measurement noise

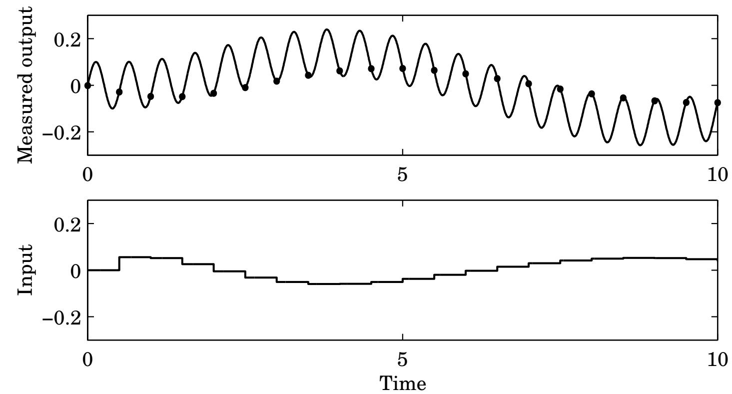
Continuous-time   Discrete-time



Sampling creates new frequencies! $\omega_{sampled} = |\omega \pm n\omega_s|$

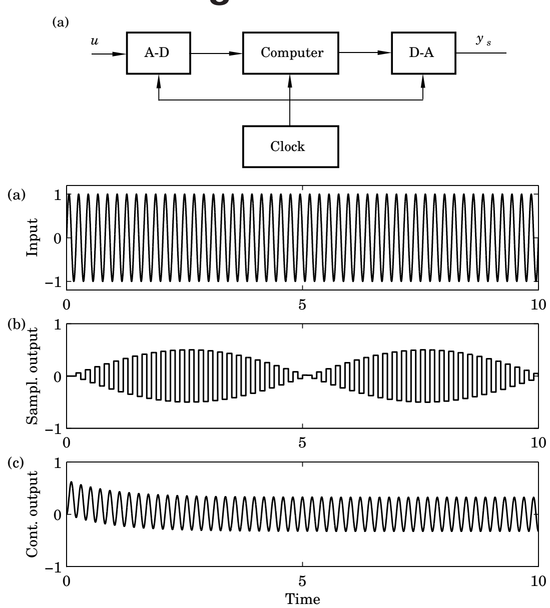
Sinusoidal measurement noise cont'd

Enlarge the scale



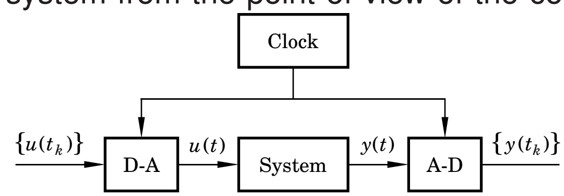
The sampling period is too long compared with the noise
Important to filter before sampling!

Problems – Higher-order harmonics



Sampling of systems

Look at the system from the point of view of the computer



- Zero-order-hold sampling of a system
- Computational issues
- Solution of the system equation
- Inverse of sampling
- Sampling of a system with time delay
- Intersample behavior

Sampling a continuous-time system

The idea:

- Let the input be piecewise constant
- Look at the sampling points only
- Use linearity and calculate step responses

System description

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Sampling a continuous-time system, cont'd

Solve the system equation (Calculate the step response, with initial value)

$$\begin{aligned} x(t) &= e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-s')}Bu(s')\,ds' \\ &= e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-s')}\,ds'\,Bu(t_k) \\ &= e^{A(t-t_k)}x(t_k) + \int_{0}^{t-t_k} e^{As}\,ds\,Bu(t_k) \\ &= \Phi(t,t_k)x(t_k) + \Gamma(t,t_k)u(t_k) \end{aligned}$$

Periodic sampling

The general case

$$x(t_{k+1}) = \Phi(t_{k+1},t_k)x(t_k) + \Gamma(t_{k+1},t_k)u(t_k)$$
$$y(t_k) = Cx(t_k) + Du(t_k)$$

$$\Phi(t_{k+1},t_k) = e^{A(t_{k+1}-t_k)} \qquad \Gamma(t_{k+1},t_k) = \int_0^{t_{k+1}-t_k} e^{As}ds\ B$$

Assume periodic sampling, i.e. $t_k = k \cdot h$, then

$$x(kh+h) = \Phi x(kh) + \Gamma u(kh)$$
$$y(kh) = Cx(kh) + Du(kh)$$

NOTE: Time-invariant linear system!

$$\Phi = e^{Ah} \qquad \Gamma = \int_0^h e^{As}\,ds\,B$$

Properties of Φ and Γ

$$\Phi = e^{Ah} \qquad \Gamma = \int_0^h e^{As}\,ds\,B$$

It thus follows that

$$\frac{d\Phi(t)}{dt} = A\Phi(t) = \Phi(t)A$$
$$\frac{d\Gamma(t)}{dt} = \Phi(t)B$$

Φ and Γ satisfy

$$\frac{d}{dt}\begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Phi(t) & \Gamma(t) \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

$\Phi(h)$ and $\Gamma(h)$ can be obtained from the block matrix

$$\begin{pmatrix} \Phi(h) & \Gamma(h) \\ 0 & I \end{pmatrix} = \exp\left\{\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} h\right\}$$

How to compute Φ and Γ?

- Numerical calculation in Matlab
- Series expansion of the matrix exponential.
- The Laplace transform of $\exp(At)$ is $(sI - A)^{-1}$.
- Cayley-Hamilton's theorem.
- Symbolic computer algebra

One way is

$$\Psi = \int_0^h e^{As}\, ds = Ih + \frac{Ah^2}{2!} + \frac{A^2h^3}{3!} + \cdots$$

The matrices Φ and Γ are given by

$$\Phi = I + A\Psi = I + Ah + \frac{(Ah)^2}{2!} + \frac{(Ah)^3}{3!} + \cdots \qquad \Gamma = \Psi B$$

Inverse of sampling

Problem: Is it always possible to find a continuous-time system that corresponds to a discrete-time system?

- Existence (Example 2.4 First order system)
- Non-uniqueness (Example 2.5 Harmonic oscillator)

Solution of the system equation

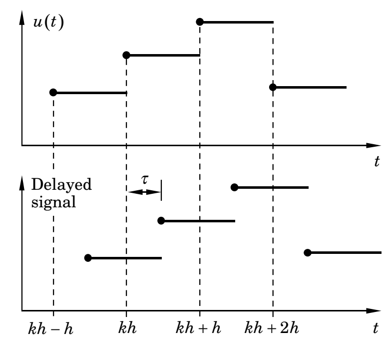
$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Iterate the equations

$$\begin{aligned} x(k) &= \Phi^{k-k_0} x(k_0) + \Phi^{k-k_0-1}\Gamma u(k_0) \\ &\quad + \cdots + \Gamma u(k-1) \\ &= \Phi^{k-k_0} x(k_0) + \sum_{j=k_0}^{k-1} \Phi^{k-j-1}\Gamma u(j) \\ y(k) &= C\Phi^{k-k_0} x(k_0) + \sum_{j=k_0}^{k-1} C\Phi^{k-j-1}\Gamma u(j) + Du(k) \\ &= C\Phi^{k-k_0} x(k_0) + \sum_{j=k_0}^{k} h(k-j)u(j) \end{aligned}$$

Initial value + Influence of input signal

Sampling of system with time delay



$$\frac{dx(t)}{dt} = Ax(t) + Bu(t-\tau)$$

$$x(kh+h) = e^{Ah}x(kh) + \int_{kh}^{kh+h} e^{A(kh+h-s')}Bu(s'-\tau)ds'$$

Time delay cont'd

Split over piece-wise constant parts

$$
\begin{aligned}
&\int_{kh}^{kh+h} e^{A(kh+h-s')} B u(s'-\tau)\, ds' \\
&= \int_{kh}^{kh+\tau} e^{A(kh+h-s')} B\, ds'\; u(kh-h) + \int_{kh+\tau}^{kh+h} e^{A(kh+h-s')} B\, ds'\; u(kh) \\
&= \Gamma_1 u(kh-h) + \Gamma_0 u(kh)
\end{aligned}
$$

$$x(kh+h) = \Phi x(kh) + \Gamma_0 u(kh) + \Gamma_1 u(kh-h)$$

where

$$\Phi = e^{Ah} \qquad \Gamma_0 = \int_0^{h-\tau} e^{As}\, ds\; B \qquad \Gamma_1 = e^{A(h-\tau)} \int_0^{\tau} e^{As}\, ds\; B$$

State-space model

$$
\begin{pmatrix} x(kh+h) \\ u(kh) \end{pmatrix} = \begin{pmatrix} \Phi & \Gamma_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(kh) \\ u(kh-h) \end{pmatrix} + \begin{pmatrix} \Gamma_0 \\ I \end{pmatrix} u(kh)
$$

Notice: r extra state variables $u(kh-h)$

Longer time delays

$$\tau = (d-1)h + \tau' \qquad 0 < \tau' \le h$$

$$
\begin{aligned}
x(kh+h) = \Phi x(kh) &+ \Gamma_0 u(kh-(d-1)h) \\
&+ \Gamma_1 u(kh-dh)
\end{aligned}
$$

Example – Double integrator with delay

Consider the double integrator with delay τ

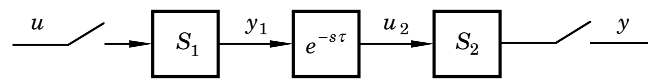
$$\Phi = e^{Ah} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
\Gamma_1 &= e^{A(h-\tau)} \int_0^{\tau} e^{As}\, ds\, B = \begin{pmatrix} 1 & h-\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\tau^2}{2} \\ \tau \end{pmatrix} \\
&= \begin{pmatrix} \tau\left(h - \frac{\tau}{2}\right) \\ \tau \end{pmatrix}
\end{aligned}
$$

$$\Gamma_0 = \int_0^{h-\tau} e^{As}\, ds\, B = \begin{pmatrix} \frac{(h-\tau)^2}{2} \\ h-\tau \end{pmatrix}$$

$$x(kh+h) = \Phi x(kh) + \Gamma_1 u(kh-h) + \Gamma_0 u(kh)$$

Inner time-delay



Let the system be described by the equations

$$
\begin{aligned}
S_1\colon \frac{dx_1(t)}{dt} &= A_1 x_1(t) + B_1 u(t) \\
y_1(t) &= C_1 x_1(t) + D_1 u(t) \\
S_2\colon \frac{dx_2(t)}{dt} &= A_2 x_2(t) + B_2 u_2(t) \\
u_2(t) &= y_1(t-\tau)
\end{aligned}
$$

$u(t)$ piecewise constant over the sampling interval h

Find a state-space description of the sampled system

Inner time-delay cont'd

Sampling with $\tau = 0$ and sampling period h

$$\begin{pmatrix} x_1(kh+h) \\ x_2(kh+h) \end{pmatrix} = \begin{pmatrix} \Phi_1(h) & 0 \\ \Phi_{21}(h) & \Phi_2(h) \end{pmatrix} \begin{pmatrix} x_1(kh) \\ x_2(kh) \end{pmatrix} + \begin{pmatrix} \Gamma_1(h) \\ \Gamma_2(h) \end{pmatrix} u(kh)$$

where

$$\Phi_i(t) = e^{A_i t}$$
$$\Gamma_1(t) = \int_0^t e^{A_1 s} B_1 \, ds$$

Wittenmark (1985), generalized in Bernhardsson (1993)

Inner time-delay – Theorem

$$\begin{aligned} x_1(kh+h) &= \Phi_1(h)x_1(kh) + \Gamma_1(h)u(kh) \\ x_2(kh+h) &= \Phi_{21}^{-}x_1(kh-h) + \Phi_2(h)x_2(kh) \\ &\quad + \Gamma_2^{-}u(kh-h) + \Gamma_2(h-\tau)u(kh) \end{aligned}$$

where

$$\Phi_i(t) = e^{A_i t} \qquad i = 1,\ 2$$
$$\Phi_{21}(t) = \int_0^t e^{A_2 s} B_2 C_1 e^{A_1(t-s)} \, ds$$
$$\Gamma_1(t) = \int_0^t e^{A_1 s} B_1 \, ds \qquad \Gamma_2(t) = \int_0^t e^{A_2 s} B_2 C_1 \Gamma_1(t-s) \, ds$$
$$\Phi_{21}^{-} = \Phi_{21}(h)\Phi_1(h-\tau)$$
$$\Gamma_2^{-} = \Phi_{21}(h)\Gamma_1(h-\tau) + \Phi_{21}(h-\tau)\Gamma_1(\tau) + \Phi_2(h-\tau)\Gamma_2(\tau)$$

Sample the delay-free system with h, $h-\tau$, and τ!

Input-output models

Solving the system equation gives

$$y(k) = C\Phi^{k-k_0}x(k_0) + \sum_{j=k_0}^{k-1} C\Phi^{k-j-1}\Gamma u(j) + Du(k)$$

Higher order difference equation
Pulse-response (weighting) function

$$h(k) = \begin{cases} 0 & k < 0 \\ D & k = 0 \\ C\Phi^{k-1}\Gamma & k \geq 1 \end{cases}$$

$$y(k) = C\Phi^{k-k_0}x(k_0) + \sum_{j=k_0}^{k} h(k-j)u(j)$$

Summary

- New phenomena
- System with time delay
- Solution of system equation
- Input-output models