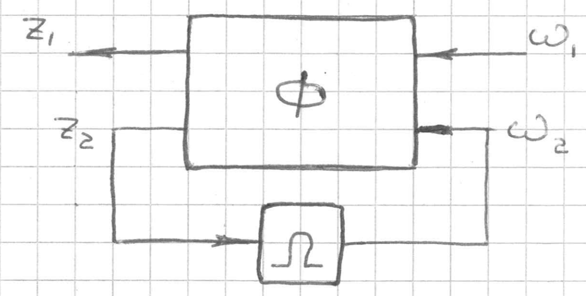
§9.1, 9.3

Linear fractional transformation (LFT)

* Lower LFT

$$\begin{cases} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \\ \\ \omega_2 = \Omega z_2 \end{cases}$$



The relation between ω_1 and z_1 can be written explicitely:

$$\boxed{\begin{aligned} F_l(\Phi, \Omega) &= \Phi_{11} + \Phi_{12}\Omega\left(1 - \Phi_{22}\Omega\right)^{-1}\Phi_{21} \\ &= \Phi_{11} + \Phi_{12}\left(1 - \Omega\Phi_{22}\right)^{-1}\Omega\Phi_{21} \end{aligned}}$$

Derivation:

$$\begin{cases} z_1 = \Phi_{11}\omega_1 + \Phi_{12}\omega_2 \\ z_2 = \Phi_{21}\omega_1 + \Phi_{22}\omega_2 \\ \omega_2 = \Omega z_2 \quad (*) \end{cases}$$

Substituting $(*)$ we get

$$\begin{cases} z_1 = \Phi_{11}\omega_1 + \Phi_{12}\Omega z_2 & (1) \\ z_2 = \Phi_{21}\omega_1 + \Phi_{22}\Omega z_2 & (2) \end{cases}$$

$$(2): \quad \left(1 - \Phi_{22}\Omega\right) z_2 = \Phi_{21}\omega_1$$

$$z_2 = \left(1 - \Phi_{22}\Omega\right)^{-1}\Phi_{21}\omega_1$$

Substituting to (1) yields

$$z_1 = \Phi_{11}\omega_1 + \Phi_{12}\Omega\left(1 - \Phi_{22}\Omega\right)^{-1}\Phi_{21}\omega_1$$

* Some properties

- Suppose C is invertible. Then

$$(A+BQ)(C+DQ)^{-1} = \mathcal{F}_l(\mathcal{M},Q)$$
$$(C+QD)^{-1}(A+QB) = \mathcal{F}_l(N,Q)$$

with

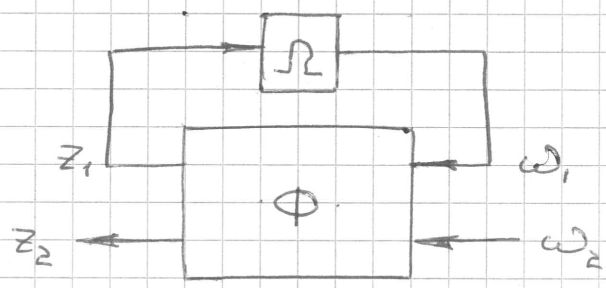
$$\mathcal{M} = \begin{bmatrix} AC^{-1} & B-AC^{-1}D \\ C^{-1} & -C^{-1}D \end{bmatrix} ; \quad N = \begin{bmatrix} C^{-1}A & C^{-1} \\ B-DC^{-1}A & -DC^{-1} \end{bmatrix}$$

- This is the reason for the name LFT

- The converse is also true if $\mathcal{M}$ and N satisfy certain conditions, see Lemma 9.2 in the book.

* Upper LFT

$$\begin{cases} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \omega_1 = \Omega z_1 \end{cases}$$



Explicit formula is:

$$\boxed{\begin{aligned} \mathcal{F}_u(\Phi,\Omega) &= \Phi_{22} + \Phi_{21}\Omega(1-\Phi_{11}\Omega)^{-1}\Phi_{12} \\ &= \Phi_{22} + \Phi_{21}(1-\Omega\Phi_{11})^{-1}\Omega\Phi_{12} \end{aligned}}$$

* Some other properties

- $$\mathcal{F}_u(\Phi, \Omega) = \mathcal{F}_l\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Phi \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \Omega\right)$$

- If $\mathcal{F}_l(\Phi, \Omega)$ is square and Φ_{11} is nonsingular, then

$$\mathcal{F}_l^{-1}(\Phi, \Omega) = \mathcal{F}_l(\hat{\Phi}, \Omega), \quad \hat{\Phi} := \begin{bmatrix} \Phi_{11}^{-1} & -\Phi_{11}^{-1}\Phi_{12} \\ \Phi_{21}\Phi_{11}^{-1} & \Phi_{22} - \Phi_{21}\Phi_{11}^{-1}\Phi_{12} \end{bmatrix}$$

- If $\mathcal{F}_u(\Phi, \Omega)$ is square and Φ_{22} is nonsingular, then

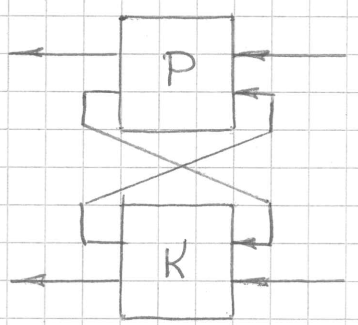
$$\mathcal{F}_u^{-1}(\Phi, \Omega) = \mathcal{F}_u(\tilde{\Phi}, \Omega), \quad \tilde{\Phi} := \begin{bmatrix} \Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21} & \Phi_{12}\Phi_{22}^{-1} \\ -\Phi_{22}^{-1}\Phi_{21} & \Phi_{22}^{-1} \end{bmatrix}$$

- If Φ is invertible, then

$$\Theta = \mathcal{F}_l(\Phi, \Omega) \iff \Omega = \mathcal{F}_u(\Phi^{-1}, \Theta)$$

* Redheffer star product

$$P \star K = \begin{bmatrix} \mathcal{F}_l(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1}K_{12} \\ K_{21}(I - P_{22}K_{11})^{-1}P_{21} & \mathcal{F}_u(K, P_{22}) \end{bmatrix}$$



- State-space formulae are provided in §9.3 in the book.
- These formulae naturally contain state-space expressions for lower and upper LFTs.

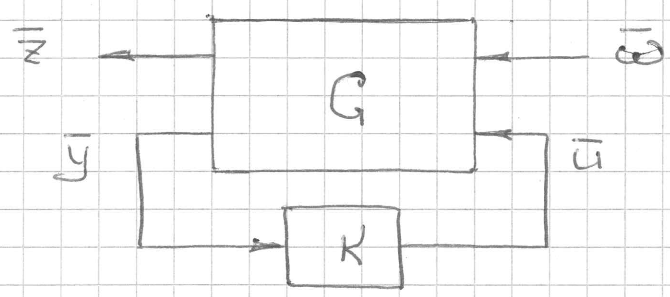
Generalized plant

Two stages of optimization-based approaches to control:

1. Formulating the control problem as a problem of minimizing a norm of certain system.
2. Solving the resulting minimization problem.

It is convenient to formulate problems in a unified fashion. (To have standard tools for the solution.)

Such a unified optimization setup is called „standard problem" or „generalize plant setup"

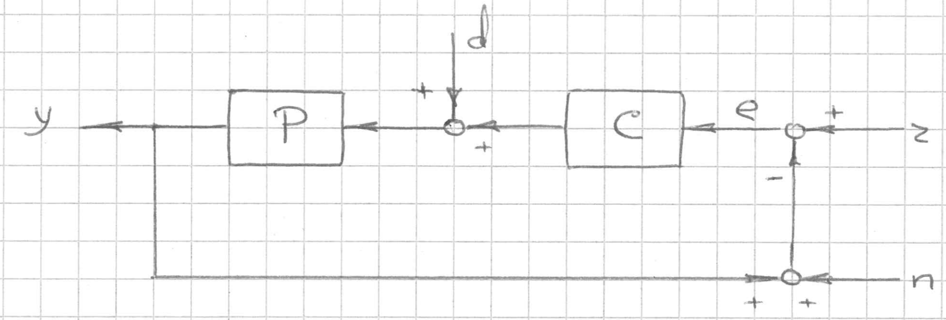


$\bar{\omega}$ - exoginous input - reference, disturbances, noises, ...
$\bar{z}$ - regulated output - tracking error, control effort, ...
$\bar{y}, \bar{u}$ - typically (but not always) measured signal and controller output.

G - generalized plant - dynamics of controlled plant, sensors, actuators; weighting functions; fixed parts of the controller.

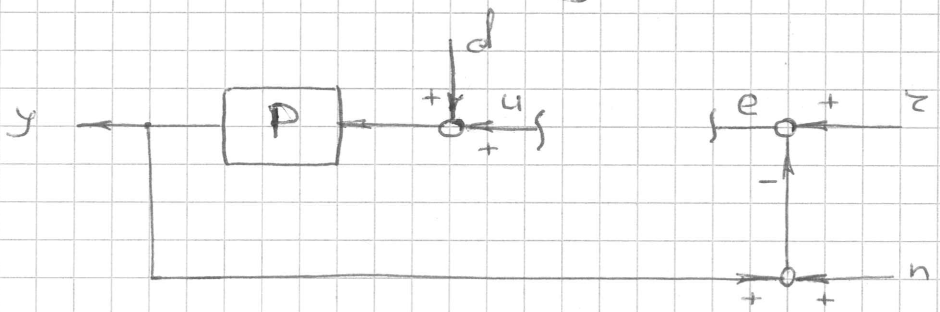
K - design parameter - part of the controller to be designed.

* Example - tracking problem



The aim is to make the tracking error e small, while keeping the control effort not too large.

To cast the problem as a generalized plant setup, we exclude the „design parameter" :



Consider $\{r, d, n, u\}$ as 4 inputs

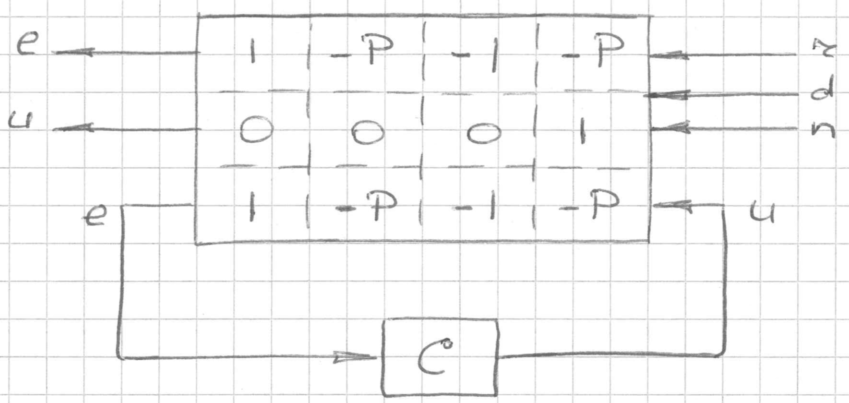
Consider $\{e, u, e\}$ as 3 outputs

- y is not included because it is not measured and not regulated.
- e is included twice because it is both measured and regulated.

It is easy to see that

$$\begin{bmatrix} e \\ u \\ e \end{bmatrix} = \begin{bmatrix} I & -P & -I & -P \\ 0 & 0 & 0 & I \\ I & -P & -I & -P \end{bmatrix} \begin{bmatrix} r \\ d \\ n \\ u \end{bmatrix}$$

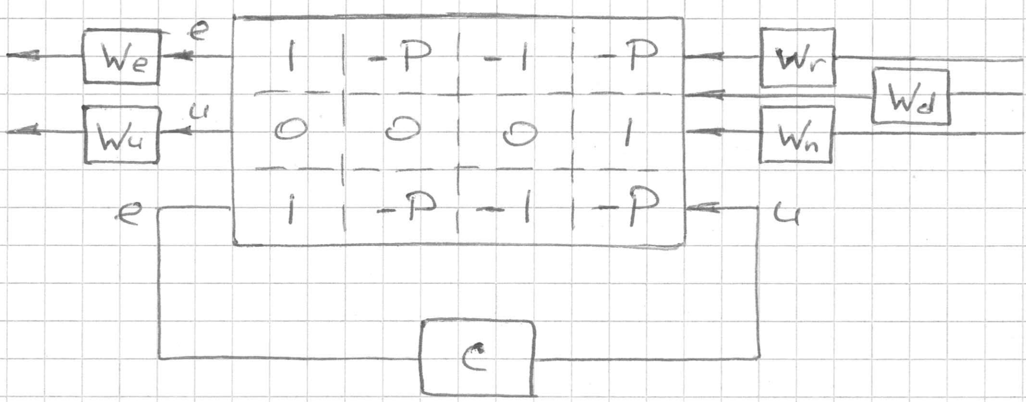
This way we get:



We would like to make e and u small, yet minimizing the norm of this system might be a nonsense!

- Impossible to make e/r and e/n small simultaniously. (Basic course in control)
- How to trade-off e and u?

We need to introduce weights:

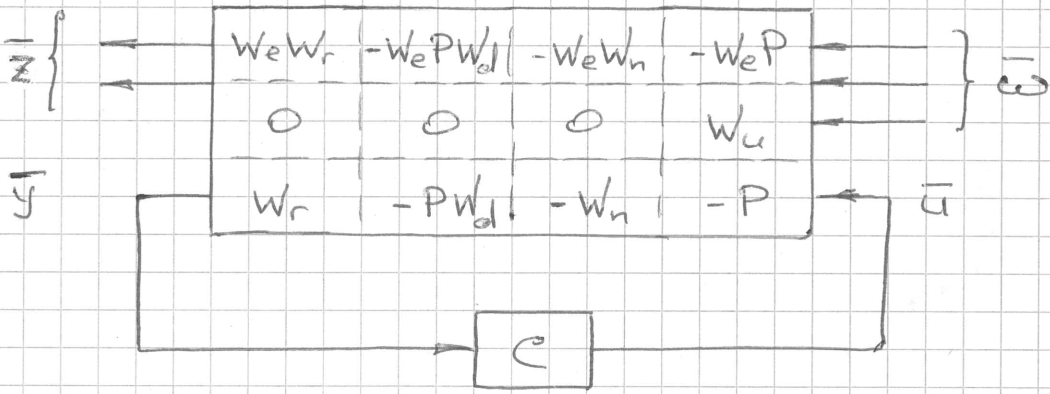


- W_r and W_d have low gains at the frequencies that are not contained in r and d and are less relevant for tracking.
(Typically low-pass filters)

- W_n has low gain at the frequencies not contained in n. (Typically high-pass filter.)

- W_e and W_u characterize trade-off between e and u in different frequencies.

Later we will talk more about the choice of the weights. Meanwhile, absorbing the weights, we get

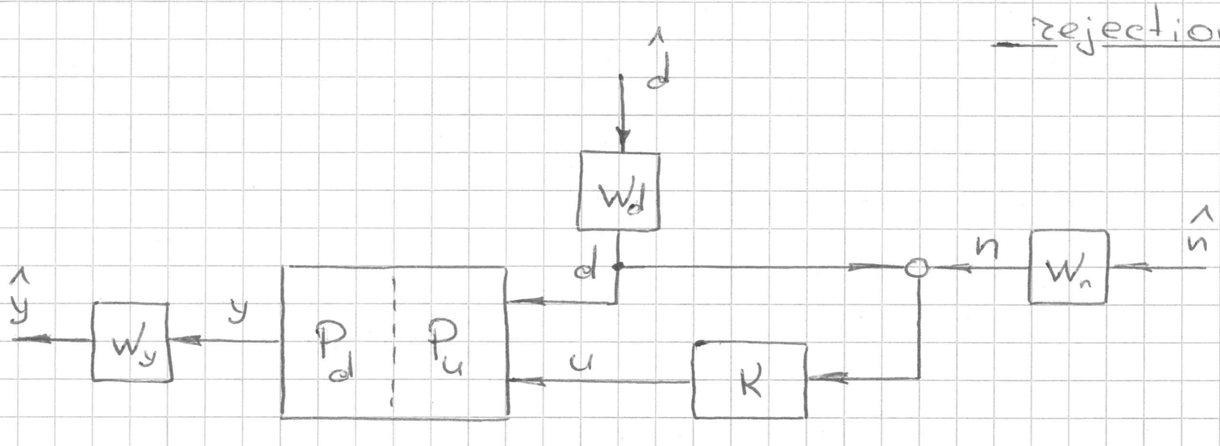


We cast the problem to finding C that guarantees internal stability (discussed later) and minimizes the norm of the system from $\bar{\omega}$ to $\bar{e}$.

So the generalized plant in this example is

$$G = \left[\begin{array}{ccc|c} W_eW_r & -W_ePW_d & -W_eW_n & -W_eP \\ 0 & 0 & 0 & W_u \\ \hline W_r & -PW_d & -W_n & -P \end{array}\right]$$

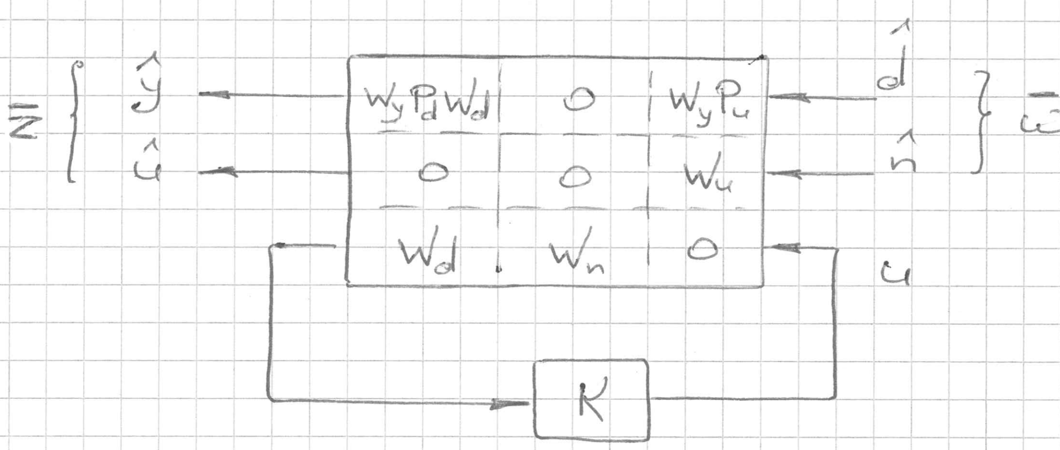
* Example – open-loop measured disturbance rejection.



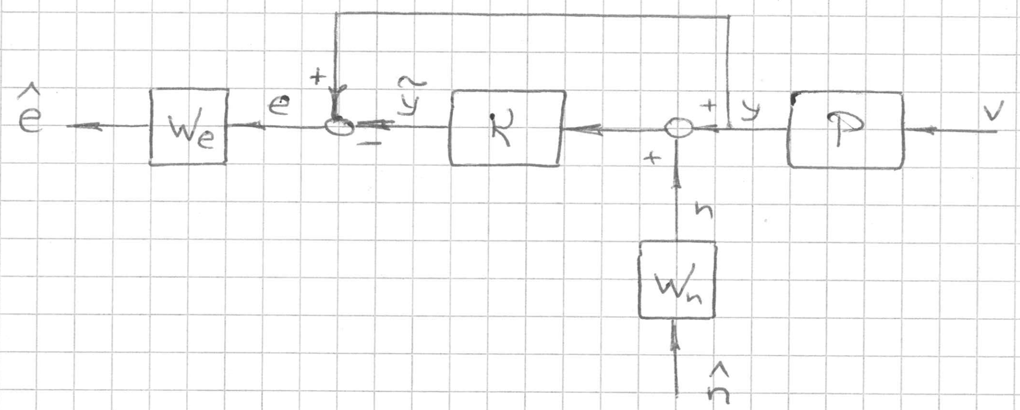
The aim is to keep y small in the presence of d, whose measurement is corrupted with noise n.

- It is convenient to introduce weights right from the beginning.

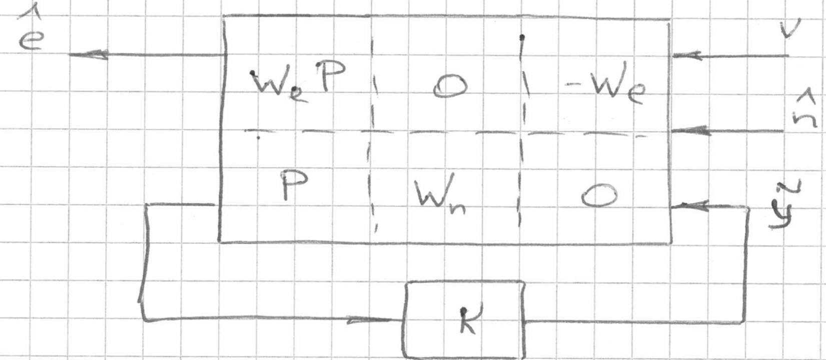
This problem can be equivalently represented by:



* Example - estimation



The aim is to estimate y with $\tilde{y} \approx y$, basing on the measurements corrupted with noise n.
In this case:



The list of examples can be continued.
After we know to formulate problems in a unified fashion, the question is how to solve the resulting standard problem.
Solutions consists of two main steps:

1. Stabilization (This lecture)
2. H_2/H_∞ optimization (Next lecture)

§5

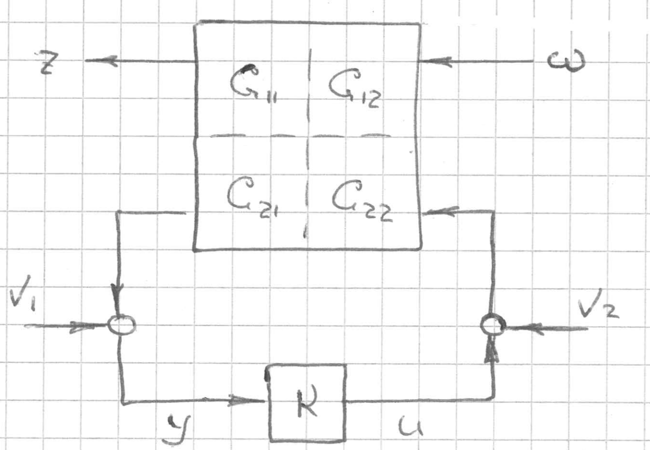
§11

Internal stability and well posedness

- We are going to study internal stability of the standard problem, described above.

- It is not enough to study stability of the system from $\bar{\omega}$ to $\bar{z}$.
  (The instabilities might be invisible through external input/output signals in case of unstable cancellation - basic course in control.)

- We will define auxiliary inputs and outputs at the interconnections:



$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = G_a \begin{bmatrix} \omega \\ v_1 \\ v_2 \end{bmatrix}$$

The system is

- well posed if all the 9 transfer matrices from ω, v_1, v_2 to z, y, u are proper.
  (well-posed <=> physically realizable)

- internally stable if all the 9 transfer matrices are stable.

* Condition for well posedness

The standard problem can be expressed by the following implicit relation:

$$\begin{bmatrix} I & 0 & -G_{12} \\ 0 & I & -G_{22} \\ 0 & -K & I \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & G_{12} \\ G_{21} & I & G_{22} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix} \qquad (*)$$

Verify this ...

Therefore it is well posed iff

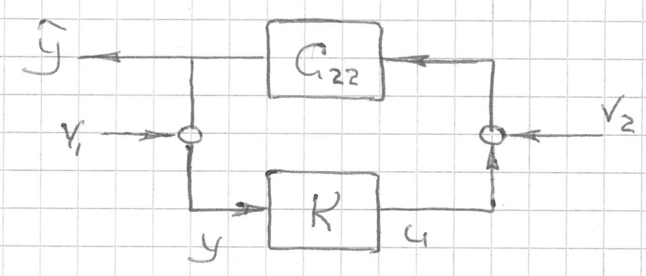
$$\begin{bmatrix} I & 0 & -G_{12}(\infty) \\ 0 & I & -G_{22}(\infty) \\ 0 & -K(\infty) & I \end{bmatrix} \quad \text{is nonsingular},$$

which is equivalent to

$$\det \begin{bmatrix} I & -G_{22}(\infty) \\ -K(\infty) & I \end{bmatrix} = \underline{\det\left(I - G_{22}(\infty)K(\infty)\right) \neq 0}$$

<u>Alternative interpretation of this condition:</u>

The feedback part of the standard problem is:



Consider, for example, $\tilde{y}/v_2 = \left(I - G_{22}(s)K(s)\right)^{-1} G_{22}(s)$.

Clearly, $\left(I - G_{22}(s)K(s)\right)$ must be invertible.

* Derivation of conditions for internal stability

- Consider the standard problem and bring in lcf for G and K:

$$\begin{cases} G = \tilde{M}^{-1}\tilde{N} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix} \\ \\ K = \tilde{M}_K^{-1}\tilde{N}_K \end{cases}$$

- Premultiplying (*) by

$$\begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} & 0 \\ \tilde{M}_{21} & \tilde{M}_{22} & 0 \\ 0 & 0 & \tilde{M}_K \end{bmatrix},$$

the implicit relation (*) can be rewritten as

$$\underbrace{\begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} & -\tilde{N}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} & -\tilde{N}_{22} \\ 0 & -\tilde{N}_K & \tilde{M}_K \end{bmatrix}}_{\tilde{M}_a} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{N}_{11} & \tilde{M}_{12} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{M}_{22} & \tilde{N}_{22} \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{N}_a} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

- It can be shown that $\tilde{M}_a^{-1}\tilde{N}_a$ is lcf for G_a.
- We have internal stability iff $\tilde{M}_a^{-1}$ is stable.

- $\tilde{M}_a^{-1}$ is stable if each column of $\tilde{M}_a$ has a stable left inverse.

- Note that the first column of $\tilde{M}_a$ depends only on G (but not on K).

- $\tilde{M}_a^{-1}$ is stable only if $\exists U, U^{-1} \in RH^\infty$, such that

$$U \begin{bmatrix} \tilde{M}_{11} \\ \tilde{M}_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- We can use U to transform lcf for G as $U \cdot \tilde{M}$, $U \cdot \tilde{N}$.

- As a result, we have stability only if there exists lcf of G having a form:

$$\tilde{M} = \begin{bmatrix} 1 & \tilde{M}_{12} \\ 0 & \tilde{M}_{22} \end{bmatrix} \qquad \tilde{N} = \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix}.$$

Note that this implies that $G_{22} = \tilde{M}_{22}^{-1} \tilde{N}_{22}$.

- So if the problem is stabilizable, we have an lcf of G_a having a form

$$\underbrace{\begin{bmatrix} 1 & \tilde{M}_{12} & -\tilde{N}_{12} \\ 0 & \tilde{M}_{22} & -\tilde{N}_{22} \\ 0 & -\tilde{N}_K & \tilde{M}_K \end{bmatrix}}_{M_a} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{N}_{11} & \tilde{M}_{12} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{M}_{22} & \tilde{N}_{22} \\ 0 & 0 & 0 \end{bmatrix}}_{N_a} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

- The new M_a is stably invertible iff

$$\begin{bmatrix} \tilde{M}_{22} & -\tilde{N}_{22} \\ -\tilde{N}_K & \tilde{M}_K \end{bmatrix}^{-1} \in RH^\infty$$

- It can be shown that there exist $K = \tilde{M}_K^{-1} \tilde{N}_K$ satisfying the condition above iff $\tilde{M}_{22}$, $\tilde{N}_{22}$ are left coprime.

Why?

* If they are not coprime, then their common zeros in RHP will be zeros of $\begin{bmatrix} \tilde{M}_{22} & \tilde{N}_{22} \\ -\tilde{N}_K & \tilde{M}_K \end{bmatrix}$

* If they are left coprime, we may assume that they are part of doubly coprime factorization of G_{22}:

$$\begin{bmatrix} X_{22} & Y_{22} \\ -\tilde{N}_{22} & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} M_{22} & -\tilde{Y}_{22} \\ N_{22} & \tilde{X}_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{M}_{22} & -\tilde{N}_{22} \\ \underset{\uparrow \atop -\tilde{N}_K}{Y_{22}} & \underset{\uparrow \atop \tilde{M}_K}{X_{22}} \end{bmatrix} \begin{bmatrix} \tilde{X}_{22} & N_{22} \\ -\tilde{Y}_{22} & M_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* Conditions for internal stability.

- The problem is stabilizable iff the generalized plant G admits lcf of a form

$$G(s) = \begin{bmatrix} I & \tilde{M}_{12} \\ 0 & \tilde{M}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix},$$

where $\tilde{M}_{22}$ and $\tilde{N}_{22}$ are left coprime, i.e., constitute lcf for $G_{22} = \tilde{M}_{22}^{-1} \tilde{N}_{22}$.

- If the problem is stabilizable and

$$\begin{bmatrix} X_{22} & Y_{22} \\ -\tilde{N}_{22} & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} M_{22} & -\tilde{Y}_{22} \\ N_{22} & \tilde{X}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G_{22} = N_{22} M_{22}^{-1} = \tilde{M}_{22}^{-1} \tilde{N}_{22}$$

is a doubly coprime factorization of G_{22}, then $K = -X_{22}^{-1} Y_{22}$ is a stabilizing controller.

- If the problem is stabilizable, then K is a stabilizing controller iff

$$\begin{bmatrix} \tilde{M}_{22} & -\tilde{N}_{22} \\ -\tilde{N}_K & \tilde{M}_K \end{bmatrix}^{-1} \in RH^\infty ,$$

where $K = \tilde{M}_K^{-1} \tilde{N}_K$ and $G_{22} = \tilde{M}_{22}^{-1} \tilde{N}_{22}$ are lcf.

<u>Another convenient representation</u>

It can be shown that

$$\begin{bmatrix} \tilde{M}_{22} & -\tilde{N}_{22} \\ -\tilde{N}_K & \tilde{M}_K \end{bmatrix}^{-1} \in RH^\infty$$

$$\Updownarrow$$

How to show this?

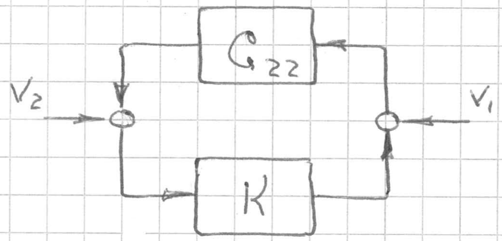
$$\begin{bmatrix} I & -G_{22} \\ -K & I \end{bmatrix}^{-1} \in RH^\infty$$

$$\Updownarrow$$

(Using Zhou p. 14)

$$\begin{bmatrix} (I-KG_{22})^{-1} & K(I-G_{22}K)^{-1} \\ G_{22}(I-KG_{22})^{-1} & (I-G_{22}K)^{-1} \end{bmatrix} \in RH^\infty$$

These are the 4 functions of the feedback part of the problem:



So if the problem is stabilizable, then

- there is no need in coprime factorization to check if K is a stabilizing controller.
- we need to check four transfer matrices.
- if $G_{22} \in RH^{\infty}$, it is enough to check $K(I - G_{22}K)^{-1}$
- if $K \in RH^{\infty}$, it is enough to check $G_{22}(I - KG_{22})^{-1}$.
- if $G_{22}, K \in RH^{\infty}$, it is enough to check $(I - G_{22}K)^{-1}$

State-space interpretation of stabilizability.

Let the generalized plant be given by minimal state-space realization:

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

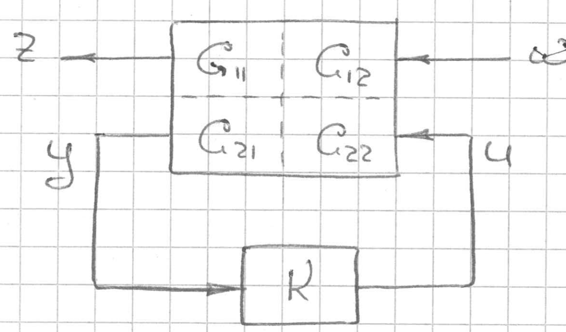
Then, the problem is stabilizable iff (A, B_2) is stabilizable and (A, C_2) is detectable.

(You will prove this at home)

§11

* Stabilization

Assume that the problem is stabilizable.
Our aim is to characterize the set of all stabilizing controllers.



Assuming stable G_{22}

- K is a stabilizing controller iff $K(1-G_{22}K)^{-1} \in RH^{\infty}$.
- Denote $Q := K(1-G_{22}K)^{-1}$
- Note that

$$Q = K(1-G_{22}K)^{-1} \quad \Big| \times (1-G_{22}K)$$

$$Q(1-G_{22}K) = K$$

$$Q = QG_{22}K + K$$

$$(1+QG_{22})^{-1}Q = K$$

- K is stabilizing iff Q is stable $\Rightarrow$

The formula above is parameterization of all stabilizing K's in terms of parameter $Q \in RH^{\infty}$.

- Let us substitute this parameterization into the relation between z and ω:

$$z/\omega = \mathcal{F}_l(G, K) = G_{11} + G_{12} \underbrace{K\left(1 - G_{22}K\right)^{-1}}_{Q} G_{21}$$

$$= G_{11} + G_{12} Q G_{21}$$

- The formula above is parameterization of all stabilized transfer matrices from ω to z.

For general G_{22}

- Introduce doubly coprime factorization of G_{22}:

$$\begin{bmatrix} X_{22} & Y_{22} \\ -\tilde{N}_{22} & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} M_{22} & -\tilde{Y}_{22} \\ N_{22} & \tilde{X}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G_{22} = N_{22} M_{22}^{-1} = \tilde{M}_{22}^{-1} \tilde{N}_{22}$$

- It can be shown (although this is very nontrivial) that K is internally stabilizing iff

$$Q := \underbrace{\left(M_{22} - K N_{22}\right)^{-1} \left(\tilde{Y}_{22} + K \tilde{X}_{22}\right)}_{\text{analogy with } K\left(1 - G_{22}K\right)^{-1} \text{ for stable } G_{22}} \in RH^{\infty}$$

- As before we can invert the formula

$$Q = \left(M_{22} - K N_{22}\right)^{-1} \left(Y_{22} + K \tilde{X}_{22}\right)$$

$$\Updownarrow$$

$$K = \left(-\tilde{Y}_{22} + M_{22} Q\right) \left(\tilde{X}_{22} + N_{22} Q\right)^{-1}$$

- The formula above parameterizes all stabilizing controllers.

- It is often called Youla/Kučera parameterization. Q is called Youla parameter.

- Using the properties of LFT the parameterization can be rewritten as:

$$K = \mathcal{F}_l(J_K, Q); \qquad J_K = \begin{bmatrix} -\hat{Y}_{22}\tilde{X}_{22}^{-1} & M_{22} + \hat{Y}_{22}\tilde{X}_{22}^{-1}N_{22} \\ \tilde{X}_{22}^{-1} & -\tilde{X}_{22}^{-1}N_{22} \end{bmatrix}$$

- Substituting this parameterization into $\mathcal{F}_l(G, K)$ yields parameterization of stabilized systems.

Summary:

Assume that the problem is stabilizable and

$$\begin{bmatrix} X_{22} & Y_{22} \\ -\tilde{N}_{22} & \tilde{M}_{22} \end{bmatrix}\begin{bmatrix} M_{22} & -\tilde{Y}_{22} \\ N_{22} & X_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G_{22} = N_{22}M_{22}^{-1} = \tilde{M}_{22}^{-1}\tilde{N}_{22}$$

- All internally stabilizing controllers can be characterized by:

$$K = \left(-\hat{Y}_{22} + M_{22}Q\right)\left(\tilde{X}_{22} + N_{22}Q\right)^{-1}$$

$$= \left(X_{22} + Q\tilde{N}_{22}\right)^{-1}\left(-Y_{22} + Q\tilde{M}_{22}\right)$$

for any $Q \in RH^{\infty}$ satisfying that either

$$\left(\tilde{X}_{22}(\infty) + N_{22}(\infty)Q(\infty)\right) \text{ or } \left(X_{22}(\infty) + Q(\infty)\tilde{N}_{22}(\infty)\right)$$

are nonsingular.

- All the corresponding stabilized transfer matrices from w to z can be characterized by:

$$T_{zw} = \underbrace{\left(G_{11} - G_{12}\tilde{Y}_{22}\tilde{M}_{22}G_{21}\right)}_{T_1} + \underbrace{G_{12}M_{22}}_{T_2} \cdot Q \cdot \underbrace{\tilde{M}_{22}G_{21}}_{T_3}$$

$$= T_1 + T_2 Q T_3$$

- Note that the expression above is affine on Q. This will play a key role in derivation of optimal solution.

State-space formulae

Given a minimal state-space realization of the generalized plant:

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

- The set of all stabilizing controllers is

$$K = \mathcal{F}_l(J_K, Q) \quad \text{for} \quad Q \in RH^\infty, \text{ where}$$

$$J_K = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & 1 \\ -(C_2 + D_{22}F) & 1 & -D_{22} \end{array}\right]$$

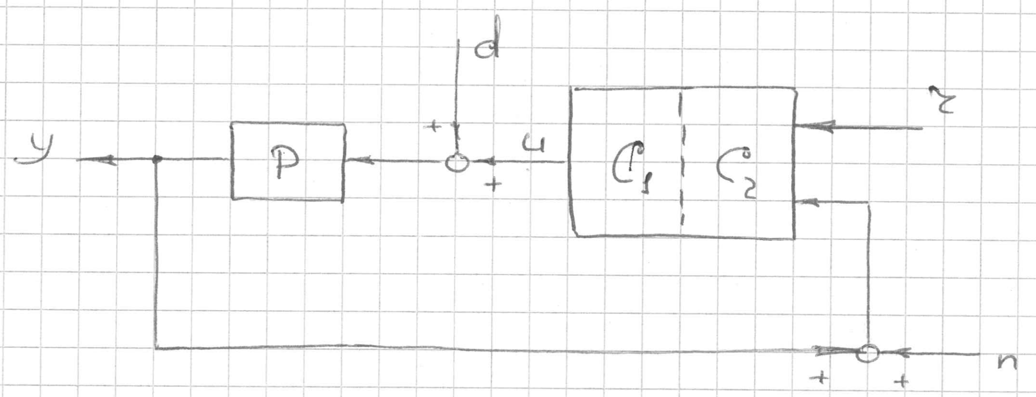
- The set of all stabilized T_{zw} is

$$T_{zw} = T_1 + T_2 Q T_3 \quad \text{for } Q \in RH^\infty, \text{ where}$$

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A + B_2 F & -B_2 F & B_1 & B_2 \\ 0 & A + L C_2 & B_1 + L D_{21} & 0 \\ \hline C_1 + D_{12} F & -D_{12} F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right]$$

2DOF control (via Youla parameterization)

Consider 2DOF control problem



and define tracking error $e := r - y$.

(For simplicity we assume no weighting functions, but introducing them wouldn't change the following reasoning.)

- Define $\bar{w} = \begin{bmatrix} d \\ n \\ r \end{bmatrix}$, $\bar{z} = \begin{bmatrix} r-y \\ u \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} r \\ y+n \end{bmatrix}$

and construct the generalized plant

$$G(s) = \left[\begin{array}{ccc|c} -P & 0 & 1 & -P \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ P & 1 & 0 & P \end{array}\right]$$

- Introduce doubly coprime factorization $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ with Bezout factors $X, Y, \tilde{X}$ and $\tilde{Y}$.

- Doubly coprime factorization for G_{22} can now be given by

$$G_{22} = \begin{bmatrix} 0 \\ P \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{M} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \tilde{N} \end{bmatrix} = \begin{bmatrix} 0 \\ N \end{bmatrix} M^{-1}$$

with

$$\left[\begin{array}{c|c} X_{22} & Y_{22} \\ \hline -\tilde{N}_{22} & \tilde{M}_{22} \end{array}\right] \left[\begin{array}{c|c} M_{22} & -\tilde{Y}_{22} \\ \hline N_{22} & \tilde{X}_{22} \end{array}\right]$$

$$= \left[\begin{array}{c|cc} X & 0 & Y \\ \hline 0 & 1 & 0 \\ -\tilde{N} & 0 & \tilde{M} \end{array}\right] \left[\begin{array}{c|cc} M & 0 & -\tilde{Y} \\ \hline 0 & 1 & 0 \\ N & 0 & \tilde{X} \end{array}\right] = I$$

- Partitioning Youla parameter $Q = [Q_1 \;\; Q_2]$ with respect to partitioning of G_{22} yields the following parameterization of stabilizing controllers:

$$C = \left(X + \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{N} \end{bmatrix} \right)^{-1} \left(-\begin{bmatrix} 0 & Y \end{bmatrix} + \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{M} \end{bmatrix} \right)$$

$$= \left(X + Q_2 \tilde{N} \right)^{-1} \begin{bmatrix} Q_1 & -Y + Q_2 \tilde{M} \end{bmatrix}$$

- The resulting parameterization of stabilized systems is

$$T_{zw} = \begin{bmatrix} -\tilde{X}\tilde{N} & 1-\tilde{X}\tilde{M} & 1 \\ -\tilde{Y}\tilde{N} & -\tilde{Y}\tilde{M} & 0 \end{bmatrix} - \begin{bmatrix} -N \\ M \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \tilde{N} & \tilde{M} & 0 \end{bmatrix}$$

$$\Downarrow$$

$$z = \left(\begin{bmatrix} -\tilde{X}\tilde{N} & 1-\tilde{X}\tilde{M} \\ -\tilde{Y}\tilde{N} & -\tilde{Y}\tilde{M} \end{bmatrix} + \begin{bmatrix} -N \\ M \end{bmatrix} Q_2 \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right) \begin{bmatrix} d \\ n \end{bmatrix}$$

$$+ \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -N \\ M \end{bmatrix} Q_1 \right) r$$

- We see that
  - response to d and n depends on Q_2 only
  - response to r depends on Q_1 only.

- The issues of (disturbance, noise) rejection and of tracking behavior are <u>independent</u>. There is <u>no trade-off</u> here!

What did we study today?

- Definition and properties of LFT.
- Generalized plant concept.
  (the standard problem)
- Conditions for internal stability
- Youla/Kučera parameterization