§4

Rational (finite-dimensional) transfer matrices.

Most of this course will be limited to systems with rational transfer matrices.
To indicate rationality, system spaces will sometimes be denoted by RL^{∞}, RH^{∞}, RL^{2} and RH^{2}.

- Rational transfer matrix is said to be proper if $G(\infty) < \infty$.

  Properness of transfer matrix <=> system causality

- Rational transfer matrix is called strictly proper if $G(\infty) = 0$ and bi-proper if $G(\infty)$ is invertible.

  Bi-properness <=> The inverse exists and is proper.

- It can be shown that:

  - RH^{∞} is a space of rational proper (causal) and stable transfer matrices
  - RH^{2} is a space of rational strictly proper and stable transfer matrices. (so $RH^{2} \subset RH^{\infty}$)

  (Note that the second statement is true for rational transfer mat., but not in general)

The rest of this lecture is dedicated to rational transfer matrices.

* <u>Is it easy to define poles and zeros for MIMO systems?</u>

Consider a SISO system $G = \dfrac{s^2+1}{s^2+s+3}$

- It has a zero at $z = \pm i$
- If $u = \sin(t)$, then $|y_{ss}| = |G(i)| \cdot 1 = 0$

Consider a MIMO system $G = \begin{bmatrix} 1 & -\dfrac{s+2}{s^2+s+3} \\ 1 & -1 \end{bmatrix}$

- Is $s = -2$ a zero? (later we will see that it is not.)
- Each of the members has no imaginary axis zeros, <u>but</u> for $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin(t)$,

$$|y_{ss}| = \left[\begin{matrix} 1 & \left| \dfrac{i+2}{i^2+i+3} \right| \\ 1 & -1 \end{matrix} \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So is there a zero at $s = \pm i$?
(later we will see that it is)

To define poles and zeros in a proper way, we will introduce Smith-McMillan form of a transfer matrix.

* Normal rank

- $\text{nrank}(G(s)) := \max_s \left(\text{rank}(G(s)) \right)$

- For all but finitely many points, $\text{nrank}(G(s)) = \text{rank}(G(s))$.

* Unimodular matrix

A square polynomial matrix $U(s)$ is called <u>unimodular</u> if $\det(U(s)) = \text{const} \neq 0$.

- The inverse of a unimodular matrix exists and is also unimodular. (and, thus, polynomial!)

* Smith - McMillan form

For any $p \times m$ transfer matrix $G(s)$ with $\text{nrank}(G(s)) = r$, there exist unimodular $U(s), V(s)$ such that

$$U(s)\,G(s)\,V(s) = \begin{bmatrix} \alpha_1(s)/\beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s)/\beta_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

where α_i divides α_{i+1}, β_{i+1} divides β_i and for each i, α_i and β_i have no common roots.

* Definitions of poles and zeros.

- The roots of the polynomial $\varphi_p(s) = \prod_{i=1}^{r} \beta_i(s)$ are called the poles.
- Multiplicities of the root in $\varphi_p(s)$ and in each of $\beta_i(s)$ are called algebraic multiplicity and partial multiplicities of the pole.
- The number of factors $\beta_i(s)$ that contain the root is called geometric multiplicity of the pole.
- The roots of the polynomial $\varphi_z(s) = \prod_{i=1}^{r} \alpha_i(s)$ are called the transmission zeros. (Usually I will shortly call them the zeros)
- Zero multiplicities are defined in the same way as for poles.
- The number $n := \deg(\varphi_p(s))$ is called McMillan degree. (Some times it is called just the order.)

Example:

$$G(s) = \begin{bmatrix} 1 & \frac{1}{s+1} \\ 0 & 1 \end{bmatrix}$$

For $U(s) = \begin{bmatrix} 1 & 0 \\ s+1 & -1 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 \\ 1 & -s-1 \end{bmatrix}$

$$U(s)\,G(s)\,V(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & s+1 \end{bmatrix}$$

So $G(s)$ has both pole and zero at $s = -1$.

Assume that s_z is not a pole of $G(s)$, then

- s_z is a transmission zero iff

$$\mathrm{rank}\left(G(s_z)\right) < \mathrm{nrank}\left(G(s)\right)$$

- Input and output directions of this zero are defined as

$$\mathrm{Ker}\left(G(s_z)\right) \quad \text{and} \quad \mathrm{Ker}\left(G'(s_z)\right)$$

Assume that s_p is not a transmission zero of invertible $G(s)$, then

- s_p is a pole iff $\mathrm{rank}\left(G^{-1}(s_p)\right) < \mathrm{nrank}\left(G^{-1}(s)\right)$

- Input and output directions of this pole are

$$\mathrm{Ker}\left(G^{-1}(s_p)'\right) \quad \text{and} \quad \mathrm{Ker}\left(G^{-1}(s_p)\right)$$

Remark: The results/definitions above can be extended for the case when S_{zp} is both a pole and a zero and $G(s)$ is not invertible ... but not in this course.

Points to think about:

- What is the direct meaning of pole direction?

- What is the role of directions in the context of pole-zero cancellations?

§ 5.4

* Coprimeness over RH^∞

Preface:

Two integers n and m are said to be coprime if their greatest common divisor is 1.

One way to verify coprimeness is by checking if there exist integers x and y such that $xm + yn = 1$.

The notion of coprimeness can be generalized for transfer matrices.

$M(s), N(s) \in RH^\infty$ are right coprime over RH^∞ if there exist $X(s), Y(s) \in RH^\infty$ such that

$$X(s)M(s) + Y(s)N(s) = I$$

(This is called „Bezout equality". $X(s)$ and $Y(s)$ are called Bezout factors.)

- $M(s)$ and $N(s)$ are right coprime <=> they do not have common RHP zeros with intersecting input directions.

  Proof of one direction:

  Assume that they do have such a zero s_z. Then there exist a vector $\eta \neq 0$ such that

  $$M(s_z)\eta = 0, \quad N(s_z)\eta = 0$$

  Post multiplying Bezout equality by η yields

  $$0 = X(s_z)M(s_z)\eta + Y(s_z)N(s_z)\eta = \eta$$

Examples:

- $M(s) = \dfrac{s+\alpha}{s+2}$, $N(s) = \dfrac{s+\alpha}{s+3}$

- $M(s) = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}$, $N(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{bmatrix}$

- $M(s) = \begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}$, $N(s) = \begin{bmatrix} \frac{s-1}{s+2} & \frac{1}{s+3} \\ 0 & \frac{s-1}{s+1} \end{bmatrix}$

$\tilde{M}(s), \tilde{N}(s) \in RH^\infty$ are left coprime over RH^∞ if there exist $\tilde{X}(s), \tilde{Y}(s) \in RH^\infty$ such that

$$\tilde{M}(s)\tilde{X}(s) + \tilde{N}(s)\tilde{Y}(s) = I$$

- Interpretation is similar to right coprimeness but for output zero directions.

- So are the examples above left coprime?

§ 5.4

* Coprime Factorization

Any proper rational transfer matrix $G(s)$ can be factorized as:

$$G(s) = \underbrace{N(s)\,M^{-1}(s)}_{\substack{\text{right coprime}\\ \text{factorization (rcf)}}} = \underbrace{\tilde{M}^{-1}(s)\,\tilde{N}(s)}_{\substack{\text{left coprime}\\ \text{factorization (lcf)}}} ,$$

where $N(s), M(s) \in RH^{\infty}$ are right coprime and $\tilde{N}(s), \tilde{M}(s) \in RH^{\infty}$ are left coprime.

Examples:

$$\underbrace{\frac{(s+1)(s-2)}{(s+3)(s-4)}}_{G(s)} =$$

$$\underbrace{\frac{s-1}{s-2}}_{G(s)} =$$

- Unstable poles of $G(s)$ coincide with the unstable poles of $M(s)$ and $\tilde{M}(s)$.
- Non minimum-phase zeros of $G(s)$ coincide with the non minimum-phase zeros of $N(s)$ and $\tilde{N}(s)$.
- Sometimes $N(s), \tilde{N}(s)$ are called numerators and $M(s), \tilde{M}(s)$ are called denominators.

- $G(s)$ is stable $\Leftrightarrow$ $\mathcal{M}^{-1}(s)$ is stable.

Proof:

Obviously, stability of $\mathcal{M}^{-1}(s)$ implies stability of $G(s)$.

On the other hand, assume $G(s)$ is stable. Then, $X\mathcal{M} + YN = I$ implies that

$$\mathcal{M}^{-1} = (X\mathcal{M} + YN)\mathcal{M}^{-1} = X + YG \in RH^{\infty}$$

- $G(s)$ is stable $\Leftrightarrow$ $\hat{\mathcal{M}}^{-1}(s)$ is stable

- Both rcf and lcf are not unique.
Let $N_1\mathcal{M}_1^{-1} = N_2\mathcal{M}_2^{-1}$ and $\tilde{\mathcal{M}}_1^{-1}\tilde{N}_1 = \tilde{\mathcal{M}}_2^{-1}\tilde{N}_2$ be rcf's and lcf's, then

$$\begin{bmatrix} \mathcal{M}_2 \\ N_2 \end{bmatrix} = \begin{bmatrix} \mathcal{M}_1 \\ N_1 \end{bmatrix} T$$

$$\begin{bmatrix} \tilde{\mathcal{M}}_2 & \tilde{N}_2 \end{bmatrix} = \tilde{T} \begin{bmatrix} \hat{\mathcal{M}}_1 & \tilde{N}_1 \end{bmatrix}$$

for some bi-stable T and $\tilde{T}$

- For any proper rational $G(s)$, doubly coprime factorization can always be constructed:

$$G(s) = N(s)\mathcal{M}^{-1}(s) = \hat{\mathcal{M}}^{-1}(s)\tilde{N}(s)$$

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{\mathcal{M}}(s) \end{bmatrix} \begin{bmatrix} \mathcal{M}(s) & -\hat{Y}(s) \\ N(s) & \tilde{X}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example:

For given $P \in RL^\infty$, characterize all $K \in RH^\infty$ that render $P \cdot K \in RH^\infty$.

Bring in rcf of P: $\begin{cases} P = N M^{-1} \\ XM + YN = I \end{cases}$

All $K \in RH^\infty$ that render $P \cdot K \in RH^\infty$ can be parameterized by:

$$K = MQ \quad , \forall Q \in RH^\infty$$

Proof:

One direction is obvious. By direct substitution we get

$$P \cdot K = P \cdot MQ = N\cancel{M^{-1}}\cancel{M}Q = NQ \in RH^\infty$$

The other direction is a bit more tricky.
Let $T = PK \in RH^\infty$ for some $K \in RH^\infty$.
Our aim is to show that this K can be expressed as $K = MQ$ for some $Q \in RH^\infty$. Namely, we need to show that $Q = M^{-1}K \in RH^\infty$.

$$T = P \cdot K$$

$$T = NM^{-1} \cdot K \quad \Big| \; Y \times \{\;\}$$

$$YT = \underbrace{YN}M^{-1} \cdot K$$

$$YT = (I - XM)M^{-1}K$$

$$YT = M^{-1}K - XK$$

$$M^{-1}K = \underbrace{YT + XK}_{Q}$$

§13.4
Z.D.G.

* Spectral factorization

- Conjugate transfer function is defined as

$$G^{\sim}(s) = G'(-s)$$

(Sometimes denoted by $G^*(s)$)

- If $G(s) = G^{\sim}(s)$, then both the sets of its poles and zeros are symmetric with respect to the imaginary axis.

- $G(s)$ that satisfies $G(s)=G^{\sim}(s)$ admits spectral factorization if there exist a bi-stable $U(s)$ such that

$$G(s) = U^{\sim}(s)\,U(s).$$

$U(s)$ is called spectral factor.

- Similarly, we define co-spectral factorization as $G(s) = V(s)V^{\sim}(s)$ with bi-stable $V(s)$.

Examples:

$$\underbrace{\frac{(s+1)(s-1)}{(s-2)(s+2)}}_{G(s)} = \frac{s-1}{s-2}\cdot\underbrace{\frac{s+1}{s+2}}_{U(s)}$$

$$\frac{s^2+1}{(s-2)(s+2)} = \text{ can not be factorized}$$

(zeros on imaginary axis)

Spectral factorization becomes somewhat less trivial in MIMO case:

$$\begin{bmatrix} \dfrac{s^2-1}{s^2-4} & \dfrac{(s-1)(2s+3)}{s^2-4} \\ \dfrac{(s+1)(2s-3)}{s^2-4} & \dfrac{5s^2-13}{s^2-4} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s-1}{s-2} & 0 \\ \dfrac{2s-3}{s-2} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{s+1}{s+2} & \dfrac{2s+3}{s+2} \\ 0 & 1 \end{bmatrix}$$

§3

* State-space realization

Rational proper transfer matrix can always be represented as:

$$G(s) = C(sI - A)^{-1}B + D$$

The input-output mapping can be described by

$$\begin{cases} \dot{x} = \mathcal{A}x + Bu \\ y = Cx + Du \end{cases}, \quad \text{with zero i.c.}$$

We will use short notation:

$$G(s) = C(sI - \mathcal{A})^{-1}B + D = \left[\begin{array}{c|c} \mathcal{A} & B \\ \hline C & D \end{array}\right]$$

Similar realizations:

Given invertible T,

$$\left[\begin{array}{c|c} \mathcal{A} & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} T\mathcal{A}T^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$$

Cancellation rules:

$$\left[\begin{array}{cc|c} \mathcal{A}_{11} & \mathcal{A}_{12} & B_1 \\ 0 & \mathcal{A}_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right] \underset{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]}{=} \left[\begin{array}{cc|c} \mathcal{A}_{22} & 0 & 0 \\ \mathcal{A}_{12} & \mathcal{A}_{11} & B_1 \\ \hline C_2 & C_1 & D \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A}_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

$$\left[\begin{array}{cc|c} \mathcal{A}_{11} & 0 & B_1 \\ \mathcal{A}_{21} & \mathcal{A}_{22} & B_2 \\ \hline C_1 & 0 & D \end{array}\right] = \left[\begin{array}{cc|c} \mathcal{A}_{22} & \mathcal{A}_{21} & B_2 \\ 0 & \mathcal{A}_{11} & B_1 \\ \hline 0 & C_1 & D \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A}_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

Why?

Minimality: Realization is minimal if there doesn't exist realization of lower order

- Order of minimal realization = Smith McMillan degree
- Any two minimal realizations are similar.

Controllability:

$\lambda \in \mathrm{spec}(A)$ is controllable from B if $[A - \lambda I \quad B]$ has full row rank.

There always exists T such that

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cc|c} A_c & * & B_c \\ 0 & A_{\bar{c}} & 0 \\ \hline C_c & C_{\bar{c}} & D \end{array}\right]$$

with the pair (A_c, B_c) controllable.

- Realization is controllable if $A_{\bar{c}}$ is void
- Realization is stabilizable if $A_{\bar{c}}$ is Hurwitz

Observability:

$\lambda \in \operatorname{spec}(A)$ is observable if $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has

full column rank.

There always exists T such that

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right] = \left[\begin{array}{cc|c} A_o & 0 & B_o \\ * & A_{\bar{o}} & B_{\bar{o}} \\ \hline C_o & 0 & D \end{array}\right]$$

with the pair (A_o, C_o) observable.

- Realization is observable if $A_{\bar{o}}$ is void.
- Realization is detectable if $A_{\bar{o}}$ is Hurwitz.

- Realization is minimal if it is controllable and observable.

-

§3

* Poles and zeros via state-space realization

Poles

- The set of poles of $G(s)$ is contained in $\mathrm{spec}(\mathcal{A})$
- For minimal realization, these sets coincide.
- The input and output pole directions are

$$B'\,\mathrm{Ker}(s_p I - \mathcal{A}') \quad \text{and} \quad C\,\mathrm{Ker}(s_p I - \mathcal{A}).$$

Invariant zeros

Invariant zeros are defined as points in which the Rosenbrock matrix

$$R(s) = \begin{bmatrix} \mathcal{A} - sI & B \\ C & D \end{bmatrix}$$

looses its normal rank.

Transmission zeros

- The set of transmission zeros is contained in the set of invariant zeros.
- For minimal realization, these sets coincide.
- The input and output zero directions are given by

$$[0 \quad I]\,\mathrm{Ker}\big(R(s_z)\big), \quad [0 \quad I]\,\mathrm{Ker}\big(R'(s_z)\big)$$

§3.4

* Basic operations in state-space

$$\left[\begin{array}{c|c} \mathcal{A}_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] + \left[\begin{array}{c|c} \mathcal{A}_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} \mathcal{A}_1 & 0 & B_1 \\ 0 & \mathcal{A}_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

$$\left[\begin{array}{c|c} \mathcal{A}_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} \mathcal{A}_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} \mathcal{A}_1 & B_1 C_2 & B_1 D_2 \\ 0 & \mathcal{A}_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]$$

$$\left[\begin{array}{c|c} \mathcal{A} & B \\ \hline C & D \end{array}\right]^{\sim} = \left[\begin{array}{c|c} -\mathcal{A}' & C' \\ \hline -B' & D' \end{array}\right]$$

$$\left[\begin{array}{c|c} \mathcal{A} & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} \mathcal{A} - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]$$

(Recall that $G(s)$ is properly invertible iff $G(\infty)$ is invertible, i.e. $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1}$ exists iff D is invertible.)

Partial fraction expansion

Assuming that $\operatorname{spec}(\mathcal{A}_1) \cap \operatorname{spec}(\mathcal{A}_2) = \emptyset$,

$$\left[\begin{array}{c|c} \mathcal{A}_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \left[\begin{array}{c|c} \mathcal{A}_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A}_1 & B_1 \\ \hline D_2 C_1 - C_2 X & 0 \end{array}\right] + \left[\begin{array}{c|c} \mathcal{A}_2 & B_2 D_1 + X B_1 \\ \hline C_2 & 0 \end{array}\right] + D_2 D_1,$$

where X satisfies

$$X \mathcal{A}_1 - \mathcal{A}_2 X + B_2 C_1 = 0$$

§4.3

* Computing H^2 norm

Given a stable strictly proper $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$

$$\|G(s)\|_2 = \sqrt{\operatorname{tr}(B'QB)} = \sqrt{\operatorname{tr}(CPC')},$$

where Q and P are defined by:

$$A'Q + QA + C'C = 0$$
$$AP + PA' + BB' = 0$$

How to calculate L^2 norm?

§4.4

* Computing H^∞ norm

There are no closed formulae.

Given a stable $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\gamma > 0$,

$\|G(s)\|_\infty < \gamma$ iff:

1. $\bar{\sigma}(D) < \gamma$

2. $$\begin{bmatrix} A + B(\gamma^2 I - D'D)^{-1} D'C & -B(\gamma^2 I - D'D)^{-1} B' \\ C'(I - \gamma^{-2} DD')^{-1} C & -(A + B(\gamma^2 I - D'D)^{-1} D'C)' \end{bmatrix}$$

   has no eigen values on imaginary axis.

The norm can be found using bisection algorithm.

* Coprime factorization via state-space

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$$\begin{bmatrix} X & Y \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & B + LD & -L \\ \hline -F & I & 0 \\ -C & -D & I \end{array}\right]$$

$$\begin{bmatrix} M & -\tilde{Y} \\ N & \tilde{X} \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B & -F \\ \hline F & I & 0 \\ C + DF & D & I \end{array}\right]$$

For any L, F guaranteing that $A+LC$, $A+BF$ Hurwitz.

(Constructing doubly coprime factorization is equivalent to solving two pole placement problems.)

§13.4 Z.D.G.

* Spectral factorization in state-space

Let $G \in RH^{\infty}$ be a right invertible transfer matrix with no zeros on the imaginary axis, given by its minimal realization:

$$G(s) = \left[\begin{array}{c|c} \mathcal{A} & B \\ \hline C & D \end{array}\right]$$

Our aim is to find spectral factor $U(s)$, satisfying

$$U^{\sim}(s)\,U(s) = G^{\sim}(s)\,G(s).$$

Derivation

Clearly, the $\mathcal{A}$-matrix of $U(s)$ should be $\mathcal{A}$ and the D-matrix should be $R^{1/2}$, where $R := D'D$.
Assume that the B-matrix is B.
Then

$$U(s) = \left[\begin{array}{c|c} \mathcal{A} & B \\ \hline \bar{C} & R^{1/2} \end{array}\right]$$

and we need just to find $\bar{C}$.
First, calculate

$$U^{\sim}U = \left[\begin{array}{c|c} -\mathcal{A}' & \bar{C}' \\ \hline -B' & R^{1/2} \end{array}\right] \left[\begin{array}{c|c} \mathcal{A} & B \\ \hline \bar{C} & R^{1/2} \end{array}\right] = \left[\begin{array}{cc|c} -\mathcal{A}' & \bar{C}'\bar{C} & \bar{C}'R^{1/2} \\ 0 & \mathcal{A} & B \\ \hline -B' & R^{1/2}\bar{C} & R \end{array}\right]$$

Similarly, derive

$$G^{\sim}G = \ldots = \left[\begin{array}{cc|c} -\mathcal{A}' & C'C & C'D \\ 0 & \mathcal{A} & B \\ \hline -B' & D'C & R \end{array}\right]$$

Apply state transformation $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ to get

$$G^{\sim}G = \left[\begin{array}{cc|c} -\mathcal{A}' & \mathcal{A}'X + X\mathcal{A} + C'C & C'D + XB \\ 0 & \mathcal{A} & B \\ \hline -B' & B'X + D'C & R \end{array}\right]$$

Comparing this to the realization of $U^{\sim}U$ yields

$$\begin{cases} \mathcal{A}'X + X\mathcal{A} + C'C = \bar{C}'\bar{C} \\ C'D + XB = \bar{C}'R^{1/2} \end{cases} \Rightarrow \bar{C}' = (C'D + XB)R^{-1/2}$$

Substituting $\bar{C}'$ to the first equation yields

$$\mathcal{A}'X + X\mathcal{A} - (C'D + XB)R^{-1}(D'C + B'X) + C'C = 0$$

This is ARE associated with

$$H := \begin{bmatrix} \mathcal{A} - BR^{-1}D'C & -BR^{-1}B' \\ -C'(I - DR^{-1}D')C & -(\mathcal{A} - BR^{-1}D'C)' \end{bmatrix}$$

<u>Final result</u>

$$U(s) = \left[\begin{array}{c|c} \mathcal{A} & B \\ \hline R^{-1/2}(D'C + B'X) & R^{1/2} \end{array}\right], \text{ where } X = \mathrm{Ric}(H).$$

<u>Remark</u>

$X = \mathrm{Ric}(H)$ exists iff

1. $(\mathcal{A}, B)$ stabilizable

2. $\begin{bmatrix} \mathcal{A} - sI & B \\ C & D \end{bmatrix}$ has full column rank on im. ax.

What did we study today?

- Poles and zeros of MIMO systems
- Coprimeness over RH^∞ and coprime factorization.
- Spectral factorization
- State-space machinery
    - Basic operations
    - Norm calculation
    - Coprime and spectral factorizations